Application of differential equation

Ex(1)

To find out time path of price :

We have the market model :

$$Q_d = a - bP \quad (a, b > 0) \longrightarrow (i)$$

$$Q_s = -c + dP \quad (c, d > 0) \longrightarrow (ii)$$

$$Q_d = Q_s \longrightarrow (iii)$$

The equilibrium price is given by

$$\bar{P} = \frac{a+c}{b+d} \text{ ——— } (iv)$$

But the equ^m price changes over time in proportion to change in excess demand and so we may not have a stable price in the short period. However the market price will be stable in the long run depending on certain conditions. The solution of differential equation enables us to obtain the conditions of dynamic stability of equilibrium.

If the initial price $P(0)$ is exactly equal to the equlibrium price $\bar{P}$, the market price is stable and there is no need of any dynamic analysis of price. But if initial price is different from the equ^m price, that is if $P(0) \neq \bar{P}$ Then market will be unstable as there will be divergence between demand and supply at a given price. Finally the price will be stabilised in due course through a process of adjustment over time, during which not only price will change but Q_d and Q_s will also change over time as they are functions of price. When the market price P_t is different from the eqm price $\bar{P}$, then the question arises — whether P_t tends to converge to $\bar{P}$ as $t \to \infty$. To test the convergence of $P_t \to \bar{P}$, we are to find out the time-path of P_t. For that we must know the pattern of price change (ie rate of price change over time)

It is well known fact in Price Theory that the change in price is governed by relative strength of demand and supply. So we can consider that the change in price over time is directly proportional to excess demand $(Q_d - Q_s)$ such that

$$\frac{dp}{dt} = \alpha(Q_d - Q_s) \quad , \alpha > 0 \longrightarrow (v)$$

$\frac{dp}{dt} \propto (Q_d - Q_s)$

$\Rightarrow \frac{dp}{dt} = \alpha(Q_d - Q_s)$

where α represents adjustment coefficient which remains constant over time.

Now, $\frac{dp}{dt} = \alpha(a - bp + c - dp)$, substituting the values of Q_d & Q_s.

$$= \alpha(a+c) - \alpha(b+d)P$$

$$\text{or, } \frac{dp}{dt} + \alpha(b+d)p = \alpha(a+c) \longrightarrow (vi)$$

This is a first order linear differential equation with constant coefficient and constant term, the solution of which consists of complementary function (Y_c) and particular integral (Y_p).

To find out the complementary f^n, we write the eqn (vi) as

$$\frac{dp}{dt} + \alpha(b+d)P = 0$$

$$\Rightarrow \frac{dp}{dt} = -\alpha(b+d)P$$

$$\Rightarrow \int \frac{dp}{P} = \int -\alpha(b+d)\,dt$$

$$\Rightarrow \log P = -\alpha(b+d)t + C$$

$$\therefore P = e^{-\alpha(b+d)t + c} = e^{c} \cdot e^{-\alpha(b+d)t} = Ae^{-\alpha(b+d)t}, \quad e^{c} = A$$

$\therefore$ Complementary f^n, $Y_c = Ae^{-\alpha(b+d)t}$

To find out particular integral Y_p, we put

$$P = \text{constant}$$

$$\Rightarrow \frac{dp}{dt} = 0$$

$\therefore$ Equation (vi) becomes

$$\alpha(b+d)P = \alpha(a+c)$$

$$\Rightarrow P = \frac{a+c}{b+d}$$

$$= \bar{P} \text{ say,}$$

= equ^n price

In equ^m,

$$q_d = q_s$$

$$\Rightarrow a - bp = -c + dp$$

$$\Rightarrow (b+d)p = a + c$$

$$\Rightarrow p = \frac{a+c}{b+d} = \bar{p} \text{ (say)}$$

The complete sol^n is, therefore,

$$P_t = Y_c + Y_p$$

$$\text{ie } P(t) = A e^{-\alpha(b+d)t} + \bar{P} \longrightarrow \text{(vii)}$$

To eliminate A, we put $t = 0$ in (vii)

$$\therefore P(0) = A e^{-\alpha(b+d).0} + \bar{P}$$

$$= A e^0 + \bar{p}$$

$$= A \cdot 1 + \bar{P}$$

$$\Rightarrow A = P(0) - \bar{P}$$

$$\therefore P(t) = [P(0) - \bar{P}] e^{-\alpha(b+d)t} + \bar{P}$$

Thus we have the time path of price as

$$P(t) = \bar{P} + (P_0 - \bar{P}) e^{-\alpha(b+d)t}$$

Now, as $t \to \infty$, ie in the long run

$$e^{-\alpha(b+d)t} \longrightarrow 0$$

$$e^{-\alpha(b+d)\infty} = e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

$$\text{Hence } \lim_{t \to \infty} P(t) = \bar{P} + 0 = \bar{P}$$

In other words, in the long run, price will converge to the equ^m price $\bar{p}$ and in this way the dynamic stability will be obtained.

In the above case the PI gives the equ^m price and c-f^n gives the deviation from the equ^m price.

[Note: The question will be like this

Demand & supply f^{n}s are given as follows

$$Q_d = a - bp \quad , a, b > 0$$
$$Q_s = -c + dp \quad , c, d > 0$$

and $\frac{dp}{dt} = \alpha (Q_d - Q_s)$

Find the time path of Price.

or

The demand and supply f^{n}s are given as.

$$Q_d = a - bp \qquad a, b > 0$$
$$Q_s = -c + dp \qquad c, d > 0$$

Assuming that the rate of change of price is proportional to the excess demand, find the time path ~~price~~ $P(t)$]

Variation: , dd equ → $Q_d = \alpha - \beta P$ $\quad \alpha, \beta > 0$

Then take the as constant of prop^l K | ss f^n → $Q_s = -\gamma + \delta P$ $\quad \gamma, \delta > 0$