

Electromagnetic Wave

Lecture 2

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Plane Electromagnetic Wave in Conducting Medium:

Maxwell's equations are

$$\operatorname{div} D = \nabla \cdot D = \rho \rightarrow (i)$$

$$\operatorname{div} B = \nabla \cdot B = 0 \rightarrow (ii)$$

$$\operatorname{curl} E = -\frac{\partial B}{\partial t} \rightarrow (iii)$$

$$\operatorname{curl} H = J + \frac{\partial D}{\partial t} \rightarrow (iv)$$

Let us assume that medium is linear and isotropic and is characterized by permittivity ϵ , permeability μ and conductivity σ , but any charge or any current other than that determined by Ohm's law.

Then

$$D = \epsilon E, B = \mu H, J = \sigma E, \rho = 0$$

So Maxwell's equation takes the form as

$$\operatorname{div} E = 0 \rightarrow (v)$$

$$\operatorname{div} H = 0 \rightarrow (vi)$$

$$\operatorname{curl} E = -\mu \frac{\partial H}{\partial t} \rightarrow (vii)$$

$$\operatorname{curl} H = \sigma E + \epsilon \frac{\partial E}{\partial t} \rightarrow (viii)$$

Taking curl of *equation (vii)* and substitute in *equation (viii)* we get

$$\text{curl curl}E = -\mu \frac{\partial}{\partial t} (\text{curl} H)$$

$$\text{curl curl}E = -\mu \frac{\partial}{\partial t} \left(\sigma E + \epsilon \frac{\partial E}{\partial t} \right)$$

$$\text{curl curl}E = -\sigma\mu \frac{\partial E}{\partial t} - \epsilon\mu \frac{\partial^2 E}{\partial t^2} \rightarrow (ix)$$

Similarly we get

$$\text{curl curl}H = -\sigma\mu \frac{\partial H}{\partial t} - \epsilon\mu \frac{\partial^2 H}{\partial t^2} \rightarrow (x)$$

Now using vector identity we get

$$\text{curl curl}A = \text{grad div}A - \nabla^2 A \rightarrow (xi)$$

We know $\text{div}E = 0$ and $\text{div}H = 0$

So equation (x) and equation (xi) becomes the form as

$$\nabla^2 E - \sigma\mu \frac{\partial E}{\partial t} - \epsilon\mu \frac{\nabla^2 E}{\partial t^2} = 0 \rightarrow (xii)$$

$$\nabla^2 H - \sigma\mu \frac{\partial H}{\partial t} - \epsilon\mu \frac{\nabla^2 H}{\partial t^2} = 0 \rightarrow (xiii)$$

These equations represents the wave equation governing electromagnetic field E and H in a homogenous isotropic conducting medium of conductivity σ .

It is apparent that these equations are vector equations of identical form which means that each of the six components of E and H separately satisfies the same scalar wave equation of the form of

$$\nabla^2 \varphi - \sigma \mu \frac{\partial \varphi}{\partial t} - \epsilon \mu \frac{\partial^2 \varphi}{\partial t^2} = 0 \rightarrow (xiv)$$

Where φ is a scalar and can stand for any one of component of E and H .

In an isotropic dielectric medium we have seen that the time varying fields are transverse i.e. the field vectors E and H are perpendicular to the direction in which the spatial variation occurs

In the limit of zero frequency we know from electrostatic and magnetostatic that the static field in a dielectric are longitudinal in the sense that the fields are derivable from scalar potential. If conductivity is not zero , modifications are necessary. Let us assume that the fields vary in inly one spatial variable x_α . Therefore decomposing the field into longitudinal and transverse parts

$$E(x_\alpha, t) = E_l(x_\alpha, t) + E_t(x_\alpha, t) \rightarrow (xv)$$

$$H(x_\alpha, t) = H_l(x_\alpha, t) + H_t(x_\alpha, t) \rightarrow (xvi)$$

Therefore

$$\frac{\partial E_l}{\partial x_\alpha} = 0 \rightarrow (xvii)$$

$$\frac{\partial H_l}{\partial x_\alpha} = 0 \rightarrow (xviii)$$

And

$$\left(\frac{\partial}{\partial t} + \frac{\sigma}{\epsilon} \right) E_l = 0 \rightarrow (xix)$$

Since

$$\text{curl} E_l = \text{curl} \text{grad} \phi_1 = 0$$

$$\frac{\partial H_l}{\partial t} = 0 \rightarrow (xx)$$

Since

$$\text{curl}H_l = \text{curl} \text{grad}\phi_2 = 0$$

From *equation (xix)* and *equation (xx)*, it is clear that longitudinal magnetic field is possible in a static uniform field. This is same situation in dielectric. But from *equation (xix)* and *equation (xx)* it seems that the longitudinal electric field is uniform in space while passes the time variation given by

$$\left(\frac{\partial}{\partial t} + \frac{\sigma}{\epsilon} \right) E_l = 0$$

$$\frac{\partial E_l}{\partial t} = -\frac{\sigma}{\epsilon} E_l$$

$$\frac{\partial E_l}{E_l} = -\frac{\sigma}{\epsilon} dt$$

Integrating this we get

$$\log E_l = -\frac{\sigma}{\epsilon} t + \log E_0$$

$$E_l(x_\alpha, t) = E_0 e^{-(\sigma/\epsilon)t} \rightarrow (xxi)$$

Consequently no static electric field can exist in a conducting medium in the absence of an applied current density. For good conductor like copper $\sigma = 10^7 \text{ mho}/\text{m}$. Therefore we shall consider the transverse field in conducting medium. Let us assume that field may vary as $e^{ik.r-i\omega t}$. Then the solution of *equation (vii)*, *equation (viii)* and *equation (ix)* are

$$E = E_0 e^{ik.r-i\omega t} \rightarrow (xxii)$$

$$H = H_0 e^{ik.r-i\omega t} \rightarrow (xxiii)$$

$$\varphi = \varphi_0 e^{ik.r-i\omega t} \rightarrow (xxiv)$$

Substituting the value of φ in the *equation (ix)* we get

$$(-k^2 + i\sigma\omega + \mu\epsilon\omega^2) = 0 \rightarrow (xxv)$$

This means that the propagation wave vector k is a complex and is given by

$$k^2 = \mu\epsilon\omega^2 \left(1 + \frac{i\sigma}{\omega\epsilon} \right) \rightarrow (xxvi)$$

In the above equation first term corresponds to displacement current and second to conduction current. As a k is complex we may write assuming σ is real. Therefore

$$k = \alpha + i\beta \rightarrow (xxvii)$$

$$k^2 = \alpha^2 - \beta^2 + 2i\alpha\beta \rightarrow (xxviii)$$

Comparing *equation (xxvi)* and *equation (xxviii)* we get

$$\alpha^2 - \beta^2 = \mu\epsilon\omega^2 \rightarrow (xxix)$$

And

$$2\alpha\beta = \mu\omega\sigma \rightarrow (xxx)$$

Where

$$\alpha = \sqrt{\mu\epsilon\omega} \left[\frac{\sqrt{\left\{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right\} + 1}}{2} \right]^{1/2} \rightarrow (xxxi)$$

$$\beta = \sqrt{\mu\epsilon\omega} \left[\frac{\sqrt{\left\{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right\} - 1}}{2} \right]^{1/2} \rightarrow (xxxii)$$

So in term of α and β , the E and H takes form of

$$E = E_0 e^{i(\alpha+i\beta)n.r-i\omega t} = E_0 e^{-\beta n.r} e^{i\alpha n.r-i\omega t} \rightarrow (xxxxii)$$

$$H = H_0 e^{i(\alpha+i\beta)n.r-i\omega t} = H_0 e^{-\beta n.r} e^{i\alpha n.r-i\omega t} \rightarrow (xxxxiii)$$

From these two equations it is obvious that field amplitude are spatially attenuated due to presence of term $e^{-\beta n.r}$. The β is measure of attenuation and known as absorption coefficient . We may conclude that the field vectors are propagated in the conduction medium with speed $v = \omega/k$ and given by

$$v = \frac{\omega}{\alpha} = \frac{1}{\sqrt{\mu\epsilon}} \left[\sqrt{\frac{\left\{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right\} + 1}{2}} \right]^{-1/2} \rightarrow (xxxxiv)$$

For poor conductor $\sigma/\omega\epsilon \ll 1$, then $\alpha = \sqrt{\mu\epsilon\omega}$, $\beta = \frac{\sigma}{2} \sqrt{\frac{\mu}{\epsilon}}$

Therefore

$$k = \alpha + i\beta = \sqrt{\mu\epsilon\omega} + i \frac{\sigma}{2} \sqrt{\frac{\mu}{\epsilon}} \rightarrow (xxxxv)$$

And for good conductor $\sigma/\omega\epsilon \gg 1$, so that α and β are approximately equal i.e.

$$\alpha = \beta = \sqrt{\mu\epsilon\omega} \sqrt{\frac{\sigma/\epsilon\omega}{2}} = \sqrt{\frac{\mu\sigma\omega}{2}}$$

Therefore

$$k = \alpha + i\beta = 1 + i\sqrt{\frac{\mu\sigma\omega}{2}} \rightarrow (xxxxvi)$$

Reflection and Refraction of Electromagnetic Wave at interface of Non-Conducting Medium:

The reflection and refraction of light at a plane surface between two media of different dielectric properties are familiar and classified into two categories,

(A) Kinematic Properties

(i) Law of reflection-The angle of reflection (θ_r) is equal to angle of incidence (θ_i)

(ii) Snell's Law say

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{n_2}{n_1}$$

Where n_1 and n_2 are indices of corresponding medium.

(iii) Law of frequency says the incident, reflected and refracted waves all have same frequency.

(iv) The incident, reflected and refracted wave all lie in same plane but normal to the boundary surface.

(B) Dynamic Properties

- (i) Intensities of reflected and refracted waves
- (ii) Phase changes and propagation

Let us consider a plane interface at $z = 0$ separating two homogenous charge free and non-conducting isotropic media characterized by permittivity's ϵ_1, ϵ_2 and permeability μ_1, μ_2

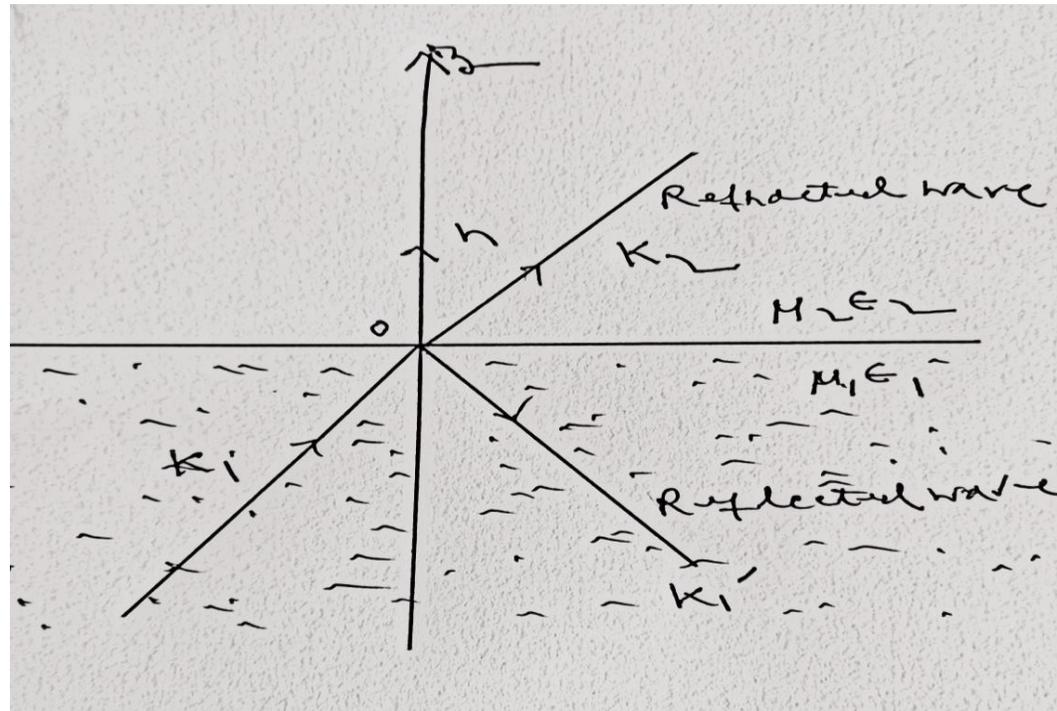


Fig 1

Let a plane wave with wave vector k_1 and frequency ω_1 be incident at point O on the interface. This wave is partly reflected and partly transmitted. Let the reflected and refracted wave vectors k'_1 and k_2 and frequencies ω'_1 and ω_2 . Let n be the unit vector to the interface. The field vector for incident, reflected and refracted wave may be;

For incident wave -

$$E_1 = E_{01} e^{ik_1 \cdot r - i\omega_1 t} \rightarrow (i)$$

$$B_1 = \frac{k_1 \times E_1}{\omega_1}$$

$$H_1 = \frac{k_1 \times E_1}{\mu_1 \omega_1} \rightarrow (ii)$$

For reflected wave -

$$E'_1 = E'_{01} e^{ik'_1 \cdot r - i\omega'_1 t} \rightarrow (iii)$$

$$B'_1 = \frac{k'_1 \times E'_1}{\omega'_1}$$

$$H'_1 = \frac{k'_1 \times E'_1}{\mu'_1 \omega'_1} \rightarrow (iv)$$

For refracted wave -

$$E_2 = E_{02} e^{ik_2 \cdot r - i\omega_2 t} \rightarrow (v)$$

$$B_2 = \frac{k_2 \times E_2}{\omega_2}$$

$$H_2 = \frac{k_2 \times E_2}{\mu_1 \omega_2} \rightarrow (vi)$$

We can apply boundary condition that the tangential component of electric field is continuous across the interface between media i.e. $z = 0$. In this case at every point in the interface

$$(E_1)_{\text{tangential}} + (E'_1)_{\text{tangential}} = (E_2)_{\text{tangential}} \rightarrow (vii)$$

$$(E_1)_{\text{tangential}} \cdot e^{ik_1 \cdot r} \cdot e^{-i\omega t} + (E'_1)_{\text{tangential}} \cdot e^{ik'_1 \cdot r} \cdot e^{-i\omega'_1 t} \\ = (E_2)_{\text{tangential}} \cdot e^{ik_2 \cdot r} \cdot e^{-i\omega_2 t} \rightarrow (viii)$$

Since this quantity is independent of time, it immediately follows that

$$\omega_1 = \omega'_1 = \omega_2 = \omega \rightarrow (ix)$$

That is the incident, reflected and refracted waves all have the same frequency. Since *equation (ix)* holds for all points of the interface $z = 0$, we must have

$$(k_1 \cdot r)_{z=0} = (k'_1 \cdot r)_{z=0} = (k_2 \cdot r)_{z=0} \rightarrow (x)$$

This equation is independent of the nature of boundary condition and contain the kinematic aspects of reflection and refraction. This *equation (x)* is independent of nature. This equation is expressed as

$$k_{1x}x + k_{1y}y = k'_{1x}x + k'_{1y}y = k_{2x}x + k_{2y}y \rightarrow (xi)$$

We get

$$k_{1x} = k'_{1x} = k_{2x} \rightarrow (xii)$$

And

$$k_{1y} = k'_{1y} = k_{2y} \rightarrow (xiii)$$

Since incident beam is in xz - plan i.e. $k_{1y} = 0$, then *equation (xiii)* becomes

$$k'_{1y} = k_{2y} \rightarrow (xiv)$$

This means k'_1 and k_2 also lie in xz – plan. As normal n is along z – axis, thus we conclude that all three wave vector and normal to the interface n all lie in the same plan. This means incident, reflected, refracted and normal to the interface all in the same plan. Further we have

$$k_1 \cdot r = k_1(x \sin \theta_i + z \cos \theta_i) \rightarrow (xv)$$

$$k'_1 \cdot r = k'_1(x \sin \theta'_r - z \cos \theta'_r) \rightarrow (xvi)$$

$$k_2 \cdot r = k_2(x \sin \theta_r + z \cos \theta_r) \rightarrow (xvii)$$

Substituting *equation (xv)* and *equation (xvi)* in *equation (xi)* we get

$$(k_1 \cdot r)_{z=0} = (k'_1 \cdot r)_{z=0}$$

We get

$$k_1 x \sin \theta_i = k'_1 x \sin \theta'_r$$

$$k_1 \sin \theta_i = k'_1 \sin \theta'_r \rightarrow (xviii)$$

Since wave vector k_1 and k'_1 lie in the same medium hence $k_1 = k'_1 = \omega\sqrt{\mu_1\epsilon_1} = \left(\frac{\omega}{v_1}\right)$, where v_1 being the phase velocity of electromagnetic wave in medium 1. So *equation (xviii)* becomes

$$\sin\theta_i = \sin\theta'_r$$

$$\theta_i = \theta'_r \rightarrow \text{(xix)}$$

i.e. the angle of incident is equal to angle of reflection. Again by putting the values of $k_1 \cdot r$ and $k_2 \cdot r$ we get

$$(k_1 \cdot r)_{z=0} = (k_2 \cdot r)_{z=0}$$

$$k_1 x \sin \theta_i = k_2 x \sin \theta_r$$

$$k_1 \sin \theta_i = k_2 \sin \theta_r$$

$$\frac{k_2}{k_1} = \frac{\sin \theta_i}{\sin \theta_r}$$

$$\frac{\sin \theta_i}{\sin \theta_r} = \frac{\omega_2 \sqrt{\mu_2 \epsilon_2}}{\omega_1 \sqrt{\mu_1 \epsilon_1}}$$

Since

$$\omega_1 = \omega_2 = \omega$$

$$\frac{\sin\theta_i}{\sin\theta_r} = \sqrt{\frac{\mu_2\epsilon_2}{\mu_1\epsilon_1}}$$

Since

$$n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} = \frac{c}{v}$$

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{n_2}{n_1} \rightarrow (xx)$$

This is known as Snell's law of reflection.