Assumptions for non-static cosmological models:

(a) There exists a cosmic time t which is orthogonal to the spatial geometry so that in spherical polar coordinates the line element may be taken as

$$ds^2 = dt^2 - e^{\mu(r,t)}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi)$$

where the coordinates r and t are so choosen that the spherical surface r = constant moves with the meterial lying on its surface. Such a coordinate system is called comoving co-ordinate system.

(b) The three dimensional spatial surfaces belonging to t = constant are locally isotropic and homogeneous, i.e. at any epoch the universe is same everywhere in space and in every direction so that the function $\mu(r,t)$ may be taken as $\mu(r,t) = f(r) + g(t)$.

(*) Derivation of Friedmann - Robertson - Walker (FRW) line - element:

The non-static spherically symmetric line-element in comoving co-ordinates is given by

$$ds^2 = dt^2 - e^{\mu}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2), \quad (1)$$

where $\mu = \mu(r,t)$ is a function of r and t only.

Due to the hypothesis of isotropy and homogeneity of the 3-space, the function $\mu(r,t)$ must be a function of the form

$$\mu(r,t) = f(r) + g(t) \longrightarrow (2)$$

where the function $f(r)$ and $g(t)$ are to be determined.

Let $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, $x^4 = t$

Then the line element (1) takes the form

$$ds^2 = g_{ij}\, dx^i dx^j$$

where

$$g_{11} = -e^{\mu}, \quad g_{22} = -e^{\mu} r^2$$

$$g_{33} = -e^{\mu} r^2 \sin^2\theta, \quad g_{44} = 1$$

and $g_{ij} = 0$ for $i \neq j$.

Therefore the christoffel symbols of second kind are given by

$$\Gamma^i_{ii} = \frac{1}{2g_{ii}} \frac{\partial g_{ii}}{\partial x^i} \longrightarrow (3)$$

$$\Gamma^i_{ij} = \frac{1}{2g_{ii}} \frac{\partial g_{ii}}{\partial x^j} \longrightarrow (4)$$

$$\Gamma^k_{ii} = -\frac{1}{2g_{kk}} \frac{\partial g_{ii}}{\partial x^k}, \quad i \neq k \longrightarrow (5)$$

$$\Gamma^k_{ij} = 0, \quad i \neq j \neq k \longrightarrow (6)$$

From (3), we have

$$\Gamma^1_{11} = \frac{1}{2g_{11}} \frac{\partial g_{11}}{\partial x^1} = \frac{1}{2(-e^{\mu})} \frac{\partial}{\partial r}(-e^{\mu})$$

$$= \frac{1}{-2e^{\mu}} (-e^{\mu}) \frac{\partial \mu}{\partial r}$$

$$= \frac{1}{2}\mu' \text{ where } \mu' = \frac{\partial \mu}{\partial r}$$

$$\Rightarrow \Gamma^1_{11} = \frac{1}{2} f' \quad \because \mu(r,t) = f(r) + g(t).$$

From (4) we have

$$\Gamma^2_{21} = \frac{1}{2g_{22}} \frac{\partial g_{22}}{\partial x^1}$$

$$= \frac{1}{-2e^{\mu} r^2} \frac{\partial}{\partial r}(-e^{\mu} r^2)$$

$$= \frac{1}{2e^{\mu} r^2}(e^{\mu}.2r + \mu' e^{\mu} r^2)$$

$$= \frac{1}{r} + \frac{1}{2} f'$$

$$\Gamma^3_{31} = \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial x^1}$$

$$= \frac{1}{-2e^{\mu} r^2 \sin^2\theta} \frac{\partial}{\partial r}(-e^{\mu} r^2 \sin^2\theta)$$

$$= \frac{1}{2e^{\mu} r^2 \sin^2\theta} e^{\mu}.2r + \mu' e^{\mu} r^2 \sin^2\theta$$

$$= \frac{1}{r} + \frac{1}{2} f'$$

$$\Gamma^3_{32} = \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial x^2}$$

$$= \frac{1}{-2e^{\mu} r^2 \sin^2\theta} \frac{\partial}{\partial \theta}(-e^{\mu} r^2 \sin^2\theta)$$

$$= \frac{1}{2e^{\mu} r^2 \sin^2\theta} e^{\mu} r^2 . 2\sin\theta\cos\theta$$

$$= \cot\theta.$$

$$\Gamma^1_{14} = \frac{1}{2g_{11}} \frac{\partial g_{11}}{\partial x^4}$$

$$= \frac{1}{2(-e^{\mu})} \frac{\partial}{\partial t}(-e^{\mu})$$

$$= \frac{1}{2e^{\mu}} e^{\mu} \frac{\partial \mu}{\partial t}$$

$$= \frac{\dot{\mu}}{2} \quad \text{where } \dot{\mu} = \frac{\partial \mu}{\partial t}$$

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$$\Rightarrow \Gamma^1_{14} = \frac{1}{2}\dot{g} \quad ; \quad \mu(r,t) = f(r) + g(t)$$

$$\Gamma^2_{24} = \frac{1}{2g_{22}} \frac{\partial g_{22}}{\partial x^4}$$

$$= \frac{1}{2(-e^{\mu} r^2)} \frac{\partial}{\partial t}(-e^{\mu} r^2)$$

$$= \frac{1}{2e^{\mu} r^2}\left(e^{\mu}\frac{\partial \mu}{\partial t}\right) r^2$$

$$= \frac{1}{2}\dot{g}$$

$$\Gamma^3_{34} = \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial x^4} = \frac{1}{2(-e^{\mu} r^2 \sin^2\theta)} \frac{\partial}{\partial t}(-e^{\mu} r^2 \sin^2\theta)$$

$$= \frac{1}{2e^{\mu} r^2 \sin^2\theta}\left(e^{\mu}\frac{\partial \mu}{\partial t}\right) r^2 \sin^2\theta$$

$$= \frac{1}{2}\dot{g}$$

From (5), we have

$$\Gamma^4_{11} = -\frac{1}{2g_{44}} \frac{\partial g_{11}}{\partial x^4} = -\frac{1}{2}\frac{\partial}{\partial t}(-e^{\mu}) = \frac{1}{2}\dot{g} e^{\mu}$$

$$\Gamma^1_{22} = -\frac{1}{2g_{11}} \frac{\partial g_{22}}{\partial x^1} = -\frac{1}{2(-e^{\mu})} \frac{\partial}{\partial r}(-e^{\mu} r^2)$$

$$= -\frac{1}{2e^{\mu}}\left(e^{\mu}.2r + e^{\mu}\mu' r^2\right)$$

$$= -\left(r + \frac{1}{2} f'(r^2)\right)$$

$$\Gamma^4_{22} = -\frac{1}{2g_{44}} \frac{\partial g_{22}}{\partial x^4}$$

$$= -\frac{1}{2}\frac{\partial}{\partial t}(-e^{\mu} r^2)$$

$$= \frac{1}{2}\dot{g} e^{\mu} r^2$$

$$\Gamma^1_{33} = -\frac{1}{2g_{11}}\frac{\partial g_{33}}{\partial x^1} = -\frac{1}{2(-e^{\mu})}\frac{\partial}{\partial r}\left(-e^{\mu}r^2\sin^2\theta\right)$$

$$= -\frac{1}{2e^{\mu}}\left(e^{\mu}2r + e^{\mu}\mu' r^2\right)\sin^2\theta$$

$$= -\left(r + \frac{1}{2}f' r^2\right)\sin^2\theta$$

$$\Gamma^2_{33} = -\frac{1}{2g_{22}}\frac{\partial g_{33}}{\partial x^2}$$

$$= -\frac{1}{2(-e^{\mu}r^2)}\frac{\partial}{\partial\theta}\left(-e^{\mu}r^2\sin^2\theta\right)$$

$$= -\frac{1}{2e^{\mu}r^2}e^{\mu}r^2 \cdot 2\sin\theta\cos\theta$$

$$= -\sin\theta\cos\theta$$

$$\Gamma^4_{33} = -\frac{1}{2g_{44}}\frac{\partial g_{33}}{\partial x^4} = -\frac{1}{2}\frac{\partial}{\partial t}\left(-e^{\mu}r^2\sin^2\theta\right)$$

$$= \frac{1}{2}\dot{g}e^{\mu}r^2\sin^2\theta$$

And the rest all zero.

Ricci tensors are given by

$$R_{ij} = -\frac{\partial\Gamma^{\alpha}_{ij}}{\partial x^{\alpha}} + \frac{\partial\Gamma^{\alpha}_{i\alpha}}{\partial x^{j}} - \Gamma^{\beta}_{ij}\Gamma^{\alpha}_{\beta\alpha} + \Gamma^{\beta}_{i\alpha}\Gamma^{\alpha}_{\beta j}$$

$$\therefore\ R_{11} = -\frac{\partial\Gamma^{\alpha}_{11}}{\partial x^{\alpha}} + \frac{\partial\Gamma^{\alpha}_{1\alpha}}{\partial x^{1}} - \Gamma^{\beta}_{11}\Gamma^{\alpha}_{\beta\alpha} + \Gamma^{\beta}_{1\alpha}\Gamma^{\alpha}_{\beta 1}$$

$$= -\frac{\partial\Gamma^1_{11}}{\partial x^1} - \frac{\partial\Gamma^4_{11}}{\partial x^4} + \frac{\partial}{\partial x^1}\left(\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}\right)$$

$$- \Gamma^1_{11}\left(\Gamma^1_{11} + \Gamma^2_{32} + \Gamma^3_{13}\right) - \Gamma^4_{11}\left(\Gamma^1_{41} + \Gamma^2_{42} + \Gamma^3_{43}\right)$$

$$+ \Gamma^1_{11}\Gamma^1_{11} + \Gamma^4_{11}\Gamma^1_{41} + \Gamma^1_{14}\Gamma^4_{11} + \Gamma^2_{12}\Gamma^2_{12} + \Gamma^3_{13}\Gamma^3_{13}$$

$$= -\frac{\partial\Gamma^4_{11}}{\partial x^4} + \frac{\partial}{\partial x^1}\left(\Gamma^2_{12} + \Gamma^3_{13}\right) - \Gamma^1_{11}\left(\Gamma^2_{12} + \Gamma^3_{13}\right)$$

$$- \Gamma^4_{11}\left(\Gamma^2_{42} + \Gamma^3_{43}\right) + \Gamma^1_{14}\Gamma^4_{11} + \Gamma^2_{12}\Gamma^2_{12} + \Gamma^3_{13}\Gamma^3_{13}$$

$$= -\frac{\partial}{\partial t}\left(-\frac{1}{2}\dot{g}e^{\mu}\right) + \frac{\partial}{\partial r}\left\{\left(\frac{1}{r}+\frac{1}{2}f'\right)+\left(\frac{1}{r}+\frac{1}{2}f'\right)\right\}$$

$$-\frac{1}{2}f'\left\{\left(\frac{1}{r}+\frac{1}{2}f'\right)+\left(\frac{1}{r}+\frac{1}{2}f'\right)\right\}$$

$$-\frac{1}{2}\dot{g}e^{\mu}\left\{\frac{1}{2}\dot{g}+\frac{1}{2}\dot{g}\right\} + \frac{1}{2}\dot{g}\left(\frac{1}{2}\dot{g}e^{\mu}\right)$$

$$+\left(\frac{1}{r}+\frac{1}{2}f'\right)\left(\frac{1}{r}+\frac{1}{2}f'\right)+\left(\frac{1}{r}+\frac{1}{2}f'\right)\left(\frac{1}{r}+\frac{1}{2}f'\right)$$

$$= -\frac{1}{2}\ddot{g}e^{\mu} - \frac{1}{2}\dot{g}^2e^{\mu} - \frac{2}{r^2} + f'' - \frac{f'}{r}$$

$$-\frac{1}{2}f'^2 - \frac{1}{2}\dot{g}^2e^{\mu} + \frac{1}{4}\dot{g}^2e^{\mu} + \frac{2}{r^2} + 2\frac{f'}{r}$$

$$+\frac{1}{2}f'^2.$$

$$= -\frac{1}{2}\ddot{g}e^{\mu} - \frac{3}{4}\dot{g}^2e^{\mu} + f'' + \frac{f'}{r}$$

$$= f'' + \frac{f'}{r} - e^{\mu}\left(\frac{1}{2}\ddot{g}+\frac{3}{4}\dot{g}^2\right).$$

$$R_{22} = -\frac{\partial\Gamma^{\alpha}_{22}}{\partial x^{\alpha}} + \frac{\partial\Gamma^{\alpha}_{2\alpha}}{\partial x^2} - \Gamma^{\beta}_{22}\Gamma^{\alpha}_{\beta\alpha} + \Gamma^{\beta}_{2\alpha}\Gamma^{\alpha}_{\beta 2}$$

$$= -\frac{\partial\Gamma^1_{22}}{\partial x^1} - \frac{\partial\Gamma^4_{22}}{\partial x^4} + \frac{\partial\Gamma^3_{23}}{\partial x^2} - \Gamma^1_{22}\left(\Gamma^1_{11}+\Gamma^2_{12}+\Gamma^3_{13}\right)$$

$$- \Gamma^4_{22}\left(\Gamma^1_{41}+\Gamma^2_{42}+\Gamma^3_{43}\right) + \Gamma^2_{21}\Gamma^1_{22}$$

$$+\Gamma^1_{22}\Gamma^2_{12} + \Gamma^4_{22}\Gamma^2_{42} + \Gamma^2_{24}\Gamma^4_{22} + \Gamma^3_{23}\Gamma^3_{23}$$

$$= -\frac{\partial\Gamma^1_{22}}{\partial x^1} - \frac{\partial\Gamma^4_{22}}{\partial x^4} + \frac{\partial\Gamma^3_{23}}{\partial x^2} - \Gamma^1_{22}\left(\Gamma^1_{11}+\Gamma^3_{13}\right) - \Gamma^4_{22}\left(\Gamma^1_{41}+\Gamma^3_{43}\right)$$

$$+\Gamma^2_{21}\Gamma^1_{22} + \Gamma^2_{24}\Gamma^4_{22} + \Gamma^3_{23}\Gamma^3_{23}$$

$$= \frac{\partial}{\partial r}\left(r+\frac{1}{2}f'r^2\right) - \frac{\partial}{\partial t}\left(\frac{1}{2}\dot{g}e^{\mu}r^2\right) + \frac{\partial}{\partial\theta}(\cot\theta)$$

$$+\left(r+\frac{1}{2}f'r^2\right)\left(\frac{1}{2}f'+\frac{1}{r}+\frac{1}{2}f'\right) - \frac{1}{2}\dot{g}e^{\mu}r^2\dot{g}$$

$$-\left(\frac{1}{r}+\frac{1}{2}f'\right)\left(r+\frac{1}{2}f'r^2\right) + \frac{1}{2}\dot{g}\cdot\frac{1}{2}\dot{g}e^{\mu}r^2$$

$$+\cot\theta\cdot\cot\theta$$

$$= 1 + \frac{1}{2} f'' r^2 + \frac{1}{2} f' \, 2r - \frac{1}{2} \ddot{g} e^{\mu} r^2 - \frac{1}{2} \dot{g}^2 e^{\mu} r^2$$
$$- \operatorname{cosec}^2\theta + r f' + 1 + \frac{1}{2} f'^2 r^2 + \frac{1}{2} r f'$$
$$- \frac{1}{2} \dot{g}^2 e^{\mu} r^2 - 1 - \frac{1}{2} r f' - \frac{1}{2} r f' - \frac{1}{4} f'^2 r^2$$
$$+ \cot^2\theta .$$

$$= \frac{1}{2} f'' r^2 + \frac{3}{2} f' r - \frac{1}{2} \ddot{g} e^{\mu} r^2 - \frac{3}{4} \dot{g}^2 e^{\mu} r^2$$
$$+ \frac{1}{4} f'^2 r^2$$

$$= r^2 \left(\frac{1}{2} f'' + \frac{1}{4} f'^2 + \frac{3 f'}{2r} - \frac{1}{2} \ddot{g} e^{\mu} - \frac{3}{4} \dot{g}^2 e^{\mu} \right)$$

$$R_{33} = - \frac{\partial \Gamma_{33}^{\alpha}}{\partial x^{\alpha}} + \frac{\partial \Gamma_{3\alpha}^{\alpha}}{\partial x^{3}} - \Gamma_{33}^{\beta} \Gamma_{\beta\alpha}^{\alpha} + \Gamma_{3\alpha}^{\beta} \Gamma_{\beta 3}^{\alpha}$$

$$= - \frac{\partial \Gamma_{33}^{1}}{\partial x^{1}} - \frac{\partial \Gamma_{33}^{2}}{\partial x^{2}} - \frac{\partial \Gamma_{33}^{4}}{\partial x^{4}} + 0 - \Gamma_{33}^{1} \left(\Gamma_{11}^{1} + \Gamma_{12}^{2} + \Gamma_{13}^{3} \right)$$
$$- \Gamma_{33}^{2} \Gamma_{23}^{3} - \Gamma_{33}^{4} \left(\Gamma_{41}^{1} + \Gamma_{42}^{2} + \Gamma_{43}^{3} \right) + \Gamma_{31}^{3} \Gamma_{33}^{1}$$
$$+ \Gamma_{33}^{1} \Gamma_{13}^{3} + \Gamma_{32}^{3} \Gamma_{33}^{2} + \Gamma_{33}^{2} \Gamma_{23}^{3} + \Gamma_{33}^{4} \Gamma_{43}^{3}$$
$$+ \Gamma_{34}^{3} \Gamma_{33}^{4}$$

$$= - \frac{\partial \Gamma_{33}^{1}}{\partial x^{1}} - \frac{\partial \Gamma_{33}^{2}}{\partial x^{2}} - \frac{\partial \Gamma_{33}^{4}}{\partial x^{4}} - \Gamma_{33}^{1} \left(\Gamma_{11}^{1} + \Gamma_{12}^{2} \right)$$
$$- \Gamma_{33}^{4} \left(\Gamma_{41}^{1} + \Gamma_{42}^{2} \right) + \Gamma_{31}^{3} \Gamma_{33}^{1} + \Gamma_{32}^{3} \Gamma_{33}^{2}$$
$$+ \Gamma_{34}^{3} \Gamma_{33}^{4}$$

$$= \frac{\partial}{\partial r} \left\{ \left(r + \frac{1}{2} f' r^2 \right) \sin^2\theta \right\} + \frac{\partial}{\partial \theta} \left(\sin\theta \cos\theta \right)$$
$$- \frac{1}{2} \frac{\partial}{\partial t} \left(\dot{g} e^{\mu} r^2 \sin^2\theta \right) + \left\{ \left(r + \frac{1}{2} f' r^2 \right) \sin^2\theta \right\} \cdot \left(\frac{1}{r} + f' \right)$$
$$- \frac{1}{2} \dot{g} e^{\mu} r^2 \sin^2\theta \cdot g - \left(\frac{1}{r} + \frac{1}{2} f' \right) \left\{ \left(r + \frac{1}{2} f' r^2 \right) \sin^2\theta \right\}$$
$$- \cot\theta \cdot \sin\theta \cos\theta + \frac{1}{4} g \cdot \dot{g} e^{\mu} r^2 \sin^2\theta$$

$$= 1 + \frac{1}{2} f'' r^2 + \frac{1}{2} f' \cdot 2r - 1 - \frac{1}{2} \ddot{g} e^{\mu} r^2$$
$$- \frac{1}{2} \dot{g}^2 e^{\mu} r^2 + r f' + 1 + \frac{1}{2} f'^2 r^2 + \frac{1}{2} r f'$$
$$- \frac{1}{2} \dot{g}^2 e^{\mu} r^2 - 1 - \frac{1}{2} r f' - \frac{1}{2} r f' - \frac{1}{4} f'^2 r^2$$
$$+ \frac{1}{4} \dot{g}^2 e^{\mu} r^2 \sin^2\theta .$$

$$= r^2 \left(\frac{1}{2} f'' + \frac{3}{2} \frac{f'}{r} - \frac{1}{2} \ddot{g} e^{\mu} - \frac{3}{4} \dot{g}^2 e^{\mu} + \frac{1}{4} f'^2 \right) \sin^2\theta .$$

$$= r^2 \left(\frac{1}{2} f'' + \frac{1}{4} f'^2 + \frac{3}{2} \frac{f'}{r} - \frac{1}{2} \ddot{g} e^{\mu} - \frac{3}{4} \dot{g}^2 e^{\mu} \right) \sin^2\theta$$

$$= R_{22} \sin^2\theta$$

$$R_{44} = -\frac{\partial \Gamma^{\alpha}_{44}}{\partial x^{\alpha}} + \frac{\partial \Gamma^{\alpha}_{4\alpha}}{\partial x^4} - \Gamma^{\beta}_{44} \Gamma^{\alpha}_{\beta\alpha} + \Gamma^{\beta}_{4\alpha} \Gamma^{\alpha}_{\beta 4}$$

$$= 0 + \frac{\partial}{\partial x^4} \Gamma^1_{41} + \Gamma^2_{42} + \Gamma^3_{43} - 0 + \Gamma^1_{41} \Gamma^1_{14} + \Gamma^2_{42} \Gamma^2_{24} + \Gamma^3_{43} \Gamma^3_{34}$$

$$= \frac{\partial}{\partial t} \left(\frac{3}{2} \dot{g} \right) + \frac{1}{2} \dot{g} \, \frac{1}{2} \dot{g} + \frac{1}{2} \dot{g} \cdot \frac{1}{2} \dot{g} + \frac{1}{2} \dot{g} \cdot \frac{1}{2} \dot{g}$$

$$= \frac{3}{2} \ddot{g} + \frac{3}{4} \dot{g}^2$$

and $R_{ij} = 0$ for $i \neq j$

Now $R^1_1 = g^{11} R_{11} = -e^{-\mu} \left[f'' + \frac{f'}{r} - e^{\mu} \left(\frac{1}{2} \ddot{g} + \frac{3}{4} \dot{g}^2 \right) \right]$

$$R^2_2 = g^{22} R_{22} = -\frac{e^{-\mu}}{r^2} \cdot r^2 \left[\frac{1}{2} f'' + \frac{1}{4} f'^2 + \frac{3}{2} \frac{f'}{r} - \frac{1}{2} \ddot{g} e^{\mu} - \frac{3}{4} \dot{g}^2 e^{\mu} \right]$$

$$= -e^{-\mu}\left[\frac{1}{2}f'' + \frac{1}{4}f'^2 + \frac{3}{2}\frac{f'}{r} - \frac{1}{2}\ddot{g}e^{\mu} - \frac{3}{4}\dot{g}^2 e^{\mu}\right]$$

$$R^3_3 = g^{33}R_{33} = -\frac{e^{-\mu}}{r^2\sin^2\theta} \cdot R_{22}\sin^2\theta$$

$$= -e^{-\mu}\left[\frac{1}{2}f'' + \frac{1}{4}f'^2 + \frac{3}{2}\frac{f'}{r} - \frac{1}{2}\ddot{g}e^{\mu} - \frac{3}{4}\dot{g}^2 e^{\mu}\right]$$

$$= R^2_2$$

$$R^4_4 = g^{44}R_{44} = \frac{3}{4}\ddot{g} + \frac{3}{4}\dot{g}^2$$

$$\therefore \quad R = R^i_j = R^1_1 + R^2_2 + R^3_3 + R^4_4$$
$$= -2e^{-\mu}\left(f'' + 2\frac{f'}{r} + \frac{1}{4}f'^2\right) + 3(\ddot{g} + \dot{g}^2)$$

Einstein's field equations are given by

$$R^i_j - \frac{1}{2}g^i_j R + \Lambda g^i_j = -8\pi T^i_j$$

$$\therefore \quad -8\pi T^1_1 = R^1_1 - \frac{1}{2}R + \Lambda$$

$$\Rightarrow 8\pi T^1_1 = -e^{-\mu}\left(\frac{f'}{r} + \frac{1}{4}f'^2\right) + \ddot{g} + \frac{3}{4}\dot{g}^2 - \Lambda \longrightarrow (7)$$

$$-8\pi T^2_2 = R^2_2 - \frac{1}{2}R + \Lambda$$

$$\Rightarrow 8\pi T^2_2 = -e^{-\mu}\left(\frac{1}{2}f'' + \frac{1}{2}\frac{f'}{r}\right) + \ddot{g} + \frac{3}{4}\dot{g}^2 - \Lambda \longrightarrow (8)$$

$$-8\pi T^3_3 = R^3_3 - \frac{1}{2}R + \Lambda \Rightarrow 8\pi T^3_3 = 8\pi T_2$$

$$\Rightarrow 8\pi T^3_3 = -e^{-\mu}\left(\frac{1}{2}f'' + \frac{1}{2}\frac{f'}{r}\right) + \ddot{g} + \frac{3}{4}\dot{g}^2 - \Lambda \longrightarrow (9)$$

$$-8\pi T^4_4 = R^4_4 - \frac{1}{2}R + \Lambda.$$

$$\Rightarrow 8\pi T^4_4 = -e^{-\mu}\left(f'' + 2\frac{f'}{r} + \frac{1}{4}f'^2\right) + \frac{3}{4}\dot{g}^2 - \Lambda \longrightarrow (10)$$

and $8\pi T^i_j = 0$ for $i \neq j$

The assumption of spatial isotropy of 3-space requires that

$$T_1^{\ 1} = T_2^{\ 2} = T_3^{\ 3}$$

$\therefore$ from (7) and (8), we have

$$8\pi T_1^{\ 1} = 8\pi T_2^{\ 2}$$

$$\Rightarrow -e^{-\mu}\left(\frac{1}{2}\beta'' - \frac{1}{2}\frac{\beta'}{r} - \frac{1}{4}\beta'^2\right) = 0$$

$$\Rightarrow \beta'' - \frac{\beta'}{r} - \frac{1}{2}\beta'^2 = 0$$

$$\Rightarrow \frac{\beta''}{\beta'} = \frac{1}{r} + \frac{\beta'}{2}$$

Integration yields

$$\log \beta' = \log r + \frac{1}{2}\beta + \log k_1$$

$$\Rightarrow \beta' = k_1 r e^{\frac{1}{2}\beta}$$

$$\Rightarrow \frac{d\beta}{dr} = k_1 r e^{\frac{1}{2}\beta},$$ where k_1 is constant of integration

$$\Rightarrow e^{-\frac{1}{2}\beta} d\beta = k_1 r\, dr$$

Integrating we get

$$\frac{e^{-\frac{1}{2}\beta}}{-\frac{1}{2}} = k_1\frac{r^2}{2} - 2k_2$$ where k_2 is a constant of integration.

$$\Rightarrow e^{-\frac{1}{2}\beta} = -k_1\frac{r^4}{4} + k_2$$

$$= k_2\left(1 - \frac{k_1}{k_2}\frac{r^2}{4}\right)$$

$$\Rightarrow e^{\beta} = \frac{k_2^{1/2}}{\left(1 - \frac{k_1}{k_2}\frac{r^2}{4}\right)^2}$$

Putting $-\frac{K_1}{K_2} = \frac{k}{R_0^2}$ $\longrightarrow$ (11)

where R_0^2 is a constant, which may be positive, negative or infinite, we can express the line-element as

$$ds^2 = dt^2 - \frac{e^{g(t)}}{\left(1 + \frac{k}{4}\frac{r^2}{R_0^2}\right)^2}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right) \longrightarrow (12)$$

where $k = +1, 0, -1$ corresponding to whether the constant R_0^2 given by (11) is positive, infinite or negative.

This completes the derivation of Friedmann - Robertson - Walker (FRW) line-element.

Dynamical consequences of FRW model:→

The Friedmann - Robertson - Walker (FRW) line-element is given by

$$ds^2 = dt^2 - e^{\mu(r,t)}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right) \longrightarrow (1)$$

with $\mu(r,t) = f(r) + g(t)$ and $e^{f(r)} = \dfrac{1}{\left(1 + \frac{K}{4}\frac{r^2}{R_0^2}\right)^2}$

where $K = +1, 0, -1$ corresponding to whether the constant R_0^2 is positive, infinite or negative.

Assuming the universe to the field with a highly ideal fluid corresponding to an isotropic and homogeneous distribution of matter having ~~averg~~ average density ρ_0 and average presure P_0, in comoving coordinate system with $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, $x^4 = t$, the energy momentum tensor T^i_j given by $T^i_j = (P_0 + \rho_0)u^i u_j - g^i_j P_0 \longrightarrow (2)$ has the component

$$\left.\begin{array}{l} T^1_1 = -P_0, \quad T^2_2 = -P_0, \quad T^3_3 = -P_0 \\ T^4_4 = \rho_0, \quad \text{and} \quad T^i_j = 0 \quad \text{for } i \neq j \end{array}\right\} \longrightarrow (3)$$

Therefore, for the line element (1) the Einstein field equations

$$R^i_j - \frac{1}{2} g^i_j R + \Lambda g^i_j = -8\pi T^i_j \longrightarrow (4)$$

gives

$$8\pi(-P_0) = -e^{-\mu}\left(\frac{f'}{r} + \frac{1}{4}f'^2\right) + \ddot{g} + \frac{3}{4}\dot{g}^2 - \Lambda \longrightarrow (5)$$

$$8\pi(-P_0) = -e^{-\mu}\left(\frac{1}{2}f'' + \frac{1}{2}\frac{f'}{r}\right) + \ddot{g} + \frac{3}{4}\dot{g}^2 - \Lambda \longrightarrow (6)$$

$$8\pi\rho_0 = -e^{-\mu}\left(f'' + 2\frac{f'}{r} + \frac{1}{4}f'^2\right) + \frac{3}{4}\dot{g}^2 - \Lambda \longrightarrow (7)$$

From (5) and (6) we have

$$-e^{-\mu}\left(\frac{1}{2}f'' - \frac{1}{2}\frac{f'}{r} - \frac{1}{4}f'^2\right) = 0$$

$$\Rightarrow f'' - \frac{f'}{r} - \frac{1}{2}f'^2 = 0$$

$$\Rightarrow f'' = \frac{f'}{r} + \frac{1}{2}f'^2 \longrightarrow (8)$$

Using (8) in (7) we get

$$8\pi \rho_0 = -e^{-\mu}\, 3\left(\frac{f'}{r} + \frac{1}{2}f'^2\right) + \frac{3}{4}\dot{g}^2 - \Lambda \longrightarrow (9)$$

But $e^{f(r)} = \dfrac{1}{\left(1+\frac{kr^2}{4R_0^2}\right)^2}$

$$\Rightarrow e^{f(r)}\left(1+\frac{kr^2}{4R_0^2}\right)^2 = 1$$

$$\Rightarrow e^{f(r)}\frac{df}{dr}\left(1+\frac{kr^2}{4R_0^2}\right)^2 + e^{f(r)}\cdot 2\left(1+\frac{kr^2}{4R_0^2}\right)\frac{k\,2r}{4R_0^2} = 0$$

$$\Rightarrow f'\left(1+\frac{kr^2}{4R_0^2}\right) + \frac{kr}{R_0^2} = 0$$

$$\Rightarrow f' = \frac{-\frac{kr}{R_0^2}}{\left(1+\frac{kr^2}{4R_0^2}\right)} \longrightarrow (10)$$

$$\therefore \frac{f'^2}{4} + \frac{f'}{r} = \frac{\frac{k^2r^2}{R_0^4}}{4\left(1+\frac{kr^2}{4R_0^2}\right)^2} + \frac{-\frac{kr}{R_0^2}}{r\left(1+\frac{kr^2}{4R_0^2}\right)}$$

$$= \frac{\frac{k}{R_0^2}}{\left(1+\frac{kr^2}{4R_0^2}\right)^2}\left[\frac{kr^2}{4R_0^2} - \left(1+\frac{kr^2}{4R_0^2}\right)\right]$$

$$= -\frac{k/R_0^2}{\left(1+\frac{kr^2}{4R_0^2}\right)^2}$$

$$\therefore e^{f(r)}\left(\frac{f'^2}{4} + \frac{f'}{r}\right) = -\frac{k}{R_0^2} \longrightarrow (11)$$

Using (11) in (9) and (5), we have

$$8\pi \rho_0 = -e^{-g(t)}\left(-\frac{3K}{R_0^2}\right) + \frac{3}{4}\dot{g}^2 - \Lambda \longrightarrow (12)$$

$$8\pi P_0 = -e^{-g(t)}\left(-\frac{K}{R_0^2}\right) - \ddot{g} - \frac{3}{4}\dot{g}^2 + \Lambda \longrightarrow (13)$$

From the equations (12) and (13), it is clear that the density ρ_0 and pressure P_0 are independent of spatial co-ordinates, but may depend on time.

Also for ρ_0 and P_0 to be positive, it is not necessary that the cosmological constant Λ be positive since we have an additional constant K and both may be adjusted to give $\rho_0 > 0$ and $P_0 \geq 0$.

Now substituting $R(t) = R_0 e^{\frac{1}{2}g(t)}$, we get

$$e^{g(t)} = \left(\frac{R}{R_0}\right)^2$$

$$\Rightarrow e^{g(t)}.\dot{g} = \frac{1}{R_0^2} 2R\dot{R}$$

$$\Rightarrow \dot{g} = 2\frac{\dot{R}}{R}$$

$$\therefore \ddot{g} = 2\frac{\ddot{R}}{R} - 2\frac{\dot{R}^2}{R^2}$$ where $\dot{R} = \frac{dR}{dt}$ and $\ddot{R} = \frac{d^2R}{dt^2}$

putting these values of $\dot{g}$ and $\ddot{g}$ in (11) and (12), we obtain

$$8\pi\rho_0 = \frac{3(\dot{R}^2+K)}{R^2} - \Lambda \longrightarrow (14)$$

$$8\pi P_0 = -\frac{2R\ddot{R}+\dot{R}^2+K}{R^2} + \Lambda \longrightarrow (15)$$

observations reveal that at the present epoch the pressure is far less than the density due to the matter.

The ration between the two quantities P_0 and ρ_0 is about 10^{-5} to 10^{-6}.

Hence neglecting P_0 from (15), we get

$$\Lambda = \frac{2R\ddot{R} + \dot{R}^2 + k}{R^2}$$

Multiplying both sides by $R^2\dot{R}$ and rearranging the terms, we get

$$(R \cdot 2\dot{R}\ddot{R} + \dot{R}^2 \cdot \dot{R}) + k\dot{R} - \Lambda R^2 \dot{R} = 0$$

Integrating we get

$$R\dot{R}^2 + kR - \Lambda \frac{R^3}{3} = \text{constant} \longrightarrow \cancel{(16)}$$

$$\Rightarrow \frac{R^2}{3}\left[\frac{3(\dot{R}^2+k)}{R^2} - \Lambda\right] = \text{constant} \longrightarrow (16)$$

comparing (16) with (14), we get

$$\frac{R^3}{3} \cdot 8\pi\rho_0 = \text{constant}$$

$$\Rightarrow \rho_0 R^3 = \text{constant}.$$

This equation represents the condition for cosmological models in which the pressure is negligibly small.

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Geometry of Friedmann - Robertson Walker (FRW) space time:

The Friedmann - Robertson - Walker (FRW) space-time is described by the line-element given by

$$ds^2 = dt^2 - \frac{e^{g(t)}}{\left(1+\frac{kr^2}{4R_0^2}\right)^2}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right) \rightarrow (1)$$

where $k = +1, 0, -1$ corresponding to whether the constant R_0^2 is positive, infinite or negative.

In order to understand the geometry of the space-time characterized by the FRW line element (1), let us consider the three cases corresponding to the three values of k, viz. $+1, 0, -1$ and rewrite the line-element in various forms.

For $k = +1$, the line-element (1) takes the form

$$ds^2 = dt^2 - \frac{e^{g(t)}}{\left(1+\frac{r^2}{4R_0^2}\right)^2}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right) \rightarrow (2)$$

Let $r_1 = \dfrac{r}{1+\frac{r^2}{4R_0^2}}$

Then $1 + \dfrac{r^2}{4R_0^2} = \dfrac{r}{r_1}$

$$\Rightarrow r_1\left(1+\frac{r^2}{4R_0^2}\right) = r$$

$$\Rightarrow dr_1 \cdot \left(1+\frac{r^2}{4R_0^2}\right) + r_1 \cdot \frac{2r\,dr}{4R_0^2} = dr$$

$$\Rightarrow dr_1\left(1+\frac{r^2}{4R_0^2}\right) = \left(1 - \frac{r r_1}{2R_0^2}\right) dr$$

$$\Rightarrow dr_1^2 = \frac{\left(1 - \frac{r r_1}{2R_0^2}\right)^2}{\left(1 + \frac{r^2}{4R_0^2}\right)^2} dr$$

Now $\left(1 - \frac{r r_1}{2R_0^2}\right)^2 = 1 - \frac{r r_1}{R_0^2} + \frac{r^2 r_1^2}{4R_0^4}$

$$= 1 - \frac{r r_1}{R_0^2} + \frac{r^2}{4R_0^2} \cdot \frac{r_1^2}{R_0^2}$$

$$= 1 - \frac{r r_1}{R_0^2} + \left(\frac{r}{r_1} - 1\right)\frac{r_1^2}{R_0^2}$$

$$= 1 - \frac{r_1^2}{R_0^2}$$

$$\therefore dr_1^2 = \frac{1 - \frac{r_1^2}{R_0^2}}{\left(1 + \frac{r^2}{4R_0^2}\right)^2} dr^2$$

$$\Rightarrow \frac{dr^2}{\left(1 + \frac{r^2}{4R_0^2}\right)^2} = \frac{dr_1^2}{1 - \frac{r_1^2}{R_0^2}}$$

Hence by substituting $r_1 = \dfrac{r}{1 + \frac{r^2}{4R_0^2}}$ in (2), we obtain,

$$ds^2 = dt^2 - e^{g(t)}\left\{\frac{dr_1^2}{1 - \frac{r_1^2}{R_0^2}} + r_1^2 d\theta^2 + r_1^2 \sin^2\theta \, d\phi^2\right\} \quad (3)$$

Which is Einstein's line-element, but now non-static.

Finally considering the transformation

$$z_1 = R_0\left(1 - \frac{r_1^2}{R_0^2}\right)^{1/2}$$

$$z_2 = r_1 \sin\theta \cos\phi$$

$$z_3 = r_1 \sin\theta \sin\phi$$

$$z_4 = r_1 \cos\theta$$

So that $z_1^2 + z_2^2 + z_3^2 + z_4^2 = R_0^2 \left(1 - \frac{r_4^2}{R_0^2}\right) +$

$$r_4^2 \sin^2\theta \cos^2\phi + r_4^2 \sin^2\theta \sin^2\phi + r_4^2 \cos^2\theta$$

$$= R_0^2 - r_4^2 + r_4^2 \left\{ \sin^2\theta \left(\cos^2\phi + \sin^2\phi\right) + \cos^2\theta \right\}$$

$$= R_0^2$$

The line-element (3) can be written as

$$ds^2 = dt^2 - e^{g(t)} \left(dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2\right) \rightarrow (5)$$

For $k=0$, the line-element (1) becomes

$$ds^2 = dt^2 - e^{g(t)} \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right) \rightarrow (6)$$

For $k=-1$, the line-element (1) takes the form

$$ds^2 = dt^2 - \frac{e^{g(t)}}{\left(1 - \frac{r^2}{4R_0^2}\right)^2} \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right) \rightarrow (7)$$

Substituting $r_4 = \frac{r}{1 - \frac{r^2}{4R_0^2}}$ in (7), so that

$$1 - \frac{r^2}{4R_0^2} = \frac{r}{r_4}$$

$$\Rightarrow r_4 \left(1 - \frac{r^2}{4R_0^2}\right) = r$$

$$\Rightarrow dr_4 \left(1 - \frac{r^2}{4R_0^2}\right) - \frac{2 r r_4}{4R_0^2} dr = dr$$

$$\Rightarrow dr_4 \left(1 - \frac{r^2}{4R_0^2}\right) = \left(1 + \frac{r r_4}{2R_0^2}\right) dr$$

$$\Rightarrow dr_4^2 = \frac{\left(1 + \frac{r r_4}{2R_0^2}\right)^2}{\left(1 - \frac{r^2}{4R_0^2}\right)^2} dr^2$$

$$\Rightarrow dr_4^2 = \frac{1 + \frac{r r_4}{R_0^2} + \frac{r^2 r_4^2}{4R_0^4}}{\left(1 - \frac{r^2}{4R_0^2}\right)^2} dr^2$$

$$\Rightarrow dr_1^2 = \frac{1 + \frac{r r_1}{R_0^2} + \frac{r_1^2}{R_0^2}\left(1 - \frac{r}{r_1}\right)}{\left(1 - \frac{r^2}{4R_0^2}\right)^2} dr^2$$

$$\Rightarrow dr_1^2 = \frac{1 + \frac{r_1^2}{R_0^2}}{\left(1 - \frac{r^2}{4R_0^2}\right)^2} dr^2$$

$$\Rightarrow \frac{dr^2}{\left(1 - \frac{r^2}{4R_0^2}\right)^2} = \frac{dr_1^2}{1 + \frac{r_1^2}{R_0^2}}$$

we obtain

$$ds^2 = dt^2 - e^{g(t)}\left\{\frac{dr_1^2}{1 + \frac{r_1^2}{R_0^2}} + r_1^2 d\theta^2 + r_1^2 \sin^2\theta\, d\phi^2\right\} \rightarrow (8)$$

Further, substituting $r_1 = R_0 \sinh \chi$ in (8), we get

$$ds^2 = dt^2 - e^{g(t)}\left\{\frac{R_0^2 \cosh^2\chi}{1 + \frac{R_0^2 \sinh^2\chi}{R_0^2}} d\chi^2 + (R_0^2 \sinh^2\chi) d\theta^2 + (R_0^2 \sinh^2\chi)\sin^2\theta\, d\phi^2\right\}$$

$$= dt^2 - R_0^2 e^{g(t)} (d\chi^2 + \sinh^2\chi\, d\theta^2 + r^2 \sinh^2\chi \sin^2\theta\, d\phi^2) \rightarrow (9)$$

From (3), (6) and (8) we observe that the FRW line-element can be written as

$$\cancel{ds^2 = dt^2 - R_0^2 e^{g(t)}(d\chi^2 +}$$

$$ds^2 = dt^2 - R^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right) \rightarrow (10)$$

where $k = +1, 0, -1$ and $R(t) = R_0 e^{\frac{1}{2}g(t)}$.

The function $R(t)$ is called the comic scale factor or simply the scale factor.

The relative expansion of the universe is parametrized by this dimensionless scale factor; therefore it is called the expansion factor.

From (10), it is obvious that when $k=+1$, r must lie between 0 and 1 but for $k=0$ and -1, r may lie between 0 and ∞.

From the line-element (4), where the angular variables are (χ, θ, ϕ), it is clear that the area of the spherical surface is $4\pi R_0^2 \sin^2\chi$ and is maximum at $\chi = \pi/2$. In this case, the entire volume of the space is finite and is given by

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} (R_0 e^{\frac{1}{2}g(t)} \sin\chi)(R_0 e^{\frac{1}{2}g(t)} \sin\chi \sin\theta)\, d\chi\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} R_0^3 e^{\frac{1}{2}g(t)} \sin^2\chi \sin\theta \, d\chi\, d\theta\, d\phi$$

$$= 2\pi^2 R_0^3 e^{\frac{3}{2}g(t)}$$

$$= 2\pi^2 R^3 \quad \text{where} \quad R = R_0 e^{\frac{1}{2}g(t)}.$$

Further the line (5) shows that the spatial geometry at any time t may be embedded in a four dimensional Euclidian space (z_1, z_2, z_3, z_4). Thus for $k=+1$, the curvature is positive and the universe is closed.

The line-element (6) represents a 3-dimensional Euclidian space such that the surface area of the spherical space is $4\pi r^2$ and the volume is infinite. The curvature of this space-time is zero and hence for $k=0$, the universe is flat.

From the line element (9), we see that surface area of the space is given by $4\pi R_0^2 \sinh^2 \chi$ which increases without limit as χ increases. This shows that we move away from the origin, the surface area increases without limit. Obviously, the volume of the space is infinite and the curvature is negative. This shows that for $k=-1$, the universe is open.

Thus the constant k in the FRW line-element describes the geometry of the spatial section of space-time with closed, flat and open universes corresponding to $k=+1, 0, -1$ respectively.

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❀ Cosmological Redshift in FRW universe:

The Friedmann-Robertson-Walker (FRW) universe is characterised by the line-element

$$ds^2 = dt^2 - R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right), \quad \rightarrow (1)$$

where $R(t)$ is the scale factor and $k=+1, 0, -1$.

Let us consider an observer located at the origin O of our coordinate system and let a light source (e.g. a nebula or a galaxy) located at any point $r=r_1$ emit two successive light pulses at times t_e and $t_e+\Delta t_e$.

Let these pulses be received by the observer at $r=0$ at times t_0 and $t_0+\Delta t_0$ respectively.

Obviously $t_0 > t_e$ and $t_0+\Delta t_0 > t_e+\Delta t_e$.

We know that the world line of a light ray is along the geodesic $ds=0$.

Therefore from (1),

$$ds=0 \Rightarrow R^2(t)\left[\frac{dr^2}{1-kr^2}+r^2 d\theta^2+r^2\sin^2\theta\, d\phi^2\right]=dt^2$$

$$\Rightarrow R^2(t)\left[\frac{1}{1-kr^2}\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{dr}{dt}\right)^2+ r^2\sin^2\theta\left(\frac{dr}{dt}\right)^2\right]=1 \longrightarrow (2)$$

Also from the symmetry of the space-time, we see that a null geodesic from $r=r_1$ to $r=0$ maintain a constant spatial direction. So considering the light ray to travel along the radial direction so that $\frac{d\theta}{dt}=0$ and $\frac{d\phi}{dt}=0$, from (2), the radial velocity of the light from or towards the origin is given by

$$\left(\frac{dr}{dt}\right)^2=\frac{1-kr^2}{R^2(t)}$$

$$\Rightarrow \frac{dr}{dt}=\pm\frac{\sqrt{1-kr^2}}{R(t)}$$

For a light ray proceeding towards the observer at rest at $r=0$, we have

$$\frac{dr}{dt}=-\frac{\sqrt{1-kr^2}}{R(t)}$$

$$\therefore \int_{t_e}^{t_0}\frac{dt}{R(t)}=-\int_{r_1}^{0}\frac{dr}{\sqrt{1-kr^2}} \longrightarrow (3)$$

And $$\int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{dt}{R(t)} = -\int_{r_1}^{0} \frac{dr}{\sqrt{1-kr^2}} \longrightarrow (4)$$

From (3) and (4), we have

$$\int_{t_e}^{t_0} \frac{dt}{R(t)} = \int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{dt}{R(t)}$$

$$\Rightarrow \int_{t_e}^{t_e+\Delta t_e} \frac{dt}{R(t)} + \int_{t_e+\Delta t_e}^{t_0} \frac{dt}{R(t)} = \int_{t_e+\Delta t_e}^{t_0} \frac{dt}{R(t)} + \int_{t_0}^{t_0+\Delta t_0} \frac{dt}{R(t)}$$

$$\Rightarrow \int_{t_e}^{t_e+\Delta t_e} \frac{dt}{R(t)} = \int_{t_0}^{t_0+\Delta t_0} \frac{dt}{R(t)}$$

$$\Rightarrow \frac{\Delta t_e}{R(t_e)} = \frac{\Delta t_0}{R(t_0)} \longrightarrow (5)$$

This shows that the time interval Δt_e of emission and Δt_0 of reception of light wave are not equal unless $R(t)$ is a constant.

Further, from the line-element (1), we see that the proper time of emision is equal to the co-ordinate time Δt_e and the proper time of reception is equal to the co-ordinate time Δt_0. Therefore $c.\Delta t_e$ is the wavelength λ_e of light wave measured by an observer at rest in the source of light and $c.\Delta t_0$ is the corresponding wave length λ_0 measured by the observer at rest at the origin of our co-ordinate system.

$$\therefore \frac{\lambda_0}{\lambda_e} = \frac{c \cdot \Delta t_0}{c \cdot \Delta t_e} \Rightarrow \frac{\lambda_0}{\lambda_e} = \frac{\Delta t_0}{\Delta t_e}$$

Using (5), we get

$$\frac{\lambda_0}{\lambda_e} = \frac{R(t_0)}{R(t_e)} \longrightarrow (6)$$

Introducing a new parameter z defined by

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e}$$

so that

$$1 + z = \frac{\lambda_0}{\lambda_e}$$

From (6), we obtain

$$1 + z = \frac{\lambda_0}{\lambda_e} = \frac{R(t_0)}{R(t_e)} \longrightarrow (7)$$

From (7) we observe that if $R(t_0) > R(t_e)$, then $\lambda_0 > \lambda_e$ and therefore, z is a positive number. This means that if $R(t_0) > R(t_e)$ then the wave length of the light wave increases by a fraction z in the transmission from the distant source to the observer at the origin of the co-ordinate system.

Consequently there is a shift towards the red end of the spectrum. In other words, the FRW model of the universe predicts a red ~~shit~~ shift. This redshift is called the cosmological redshift.

Conversely if there is a redshift,

then $r_0 > r_e$ and hence from (7), we get

$$R(t_0) > R(t_e).$$

This implies that $R(t)$ is an increasing function of time. This is in agreement with Hubble's discovery of the uniform expansion of the universe.