Class III

Case II: Roots of indicial equation are equal

Ex. 1. Solve in series about $x = 0$

$$(x - x^2)y'' + (1 - 5x)y' - 4y = 0$$

Solution:

Here x = 0 is a regular singular point.

Let $y = \sum_{n=0}^{\infty} C_n x^{n+r}, C_0 \neq 0$ be the series solution. Then after rearrangement, we get

$$\Rightarrow \sum_{n=0}^{\infty}(n+r)(n+r-1)C_n x^{n+r-1} - \sum_{n=1}^{\infty}(n+r-1)(n+r-2)C_{n-1}x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)C_n x^{n+r-1} - 5\sum_{n=1}^{\infty}(n+r-1)C_{n-1}x^{n+r-1} - 4\sum_{n=1}^{\infty} C_{n-1}x^{n+r-1} = 0 \ldots\ldots\ldots\ldots..(1)$$

The indicial equation is,

$$[r(r-1) + r]C_0 = 0$$
$$\Rightarrow r^2 C_0 = 0$$
$$\Rightarrow r = 0\ (twice)$$

The recurrence relation is,

$$(n+r)(n+r-1)C_n + (n+r)C_n - [(n+r-1)(n+r-2) + 5(n+r-1) + 4]C_{n-1} = 0, n \geq 1$$
$$\Rightarrow (n+r)^2 C_n - (n+r+1)^2 C_{n-1} = 0, n \geq 1 \ldots\ldots\ldots(2)$$
$$\Rightarrow C_n = \frac{(n+r+1)^2 C_{n-1}}{(n+r)^2}, n \geq 1 \ldots\ldots(3)$$

Now for n =1, 2, 3,……

$$C_1 = \frac{(2+r)^2 C_0}{(r+1)^2}, C_2 = \frac{(3+r)^2 C_0}{(1+r)^2}, C_3 = \frac{(5+r)^2 C_0}{(1+r)^2} \ldots\ldots\ldots$$

The assumed series solution is

$$y = x^r(C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + C_6x^6 \ldots\ldots)$$
$$y = x^r C_0\left(1 + \frac{(2+r)^2}{(r+1)^2}x + \frac{(3+r)^2}{(r+1)^2}x^2 + \frac{(5+r)^2}{(r+1)^2}x^3 \ldots\ldots.\right) \ldots(4)$$

Now, the first series solution for r =0 is,

$$y_1 = C_0(1 + 2^2x + 3^2x^2 + \cdots\ldots)$$

To find the second series solution, we proceed as follow:
The left hand side of eq. (1) is equal to

$$r^2x^{r-1} + \sum_{n=1}^{\infty}[(n+r)^2C_n - (n+r+1)^2C_{n-1}]x^{n+r-1}$$

Here, due to the recurrence relation (2), the second term of above is zero.
Therefore,

$$(x-x^2)y'' + (1-5x)y' - 4y = r^2x^{r-1}$$

$$\Rightarrow \frac{\partial}{\partial r}\{(x-x^2)y'' + (1-5x)y' - 4y\} = \frac{\partial}{\partial r}\{r^2x^{r-1}\}$$

$$\Rightarrow \left[(x-x^2)\frac{d^2}{dx^2} + (1-5x)\frac{d}{dx} - 4\right]\frac{\partial y}{\partial r} = 2rx^{r-1} + r^2x^{r-1}logx$$

$$\Rightarrow \left[(x-x^2)\frac{d^2}{dx^2} + (1-5x)\frac{d}{dx} - 4\right]\left(\frac{\partial y}{\partial r}\right)_{r=0} = 0$$

which shows that $\left(\frac{\partial y}{\partial r}\right)_{r=0}$ is also another series solution.
Now, from (4), we have,

$$\frac{\partial y}{\partial r} = x^rC_0logx\left(1 + \frac{(2+r)^2}{(r+1)^2}x + \frac{(3+r)^2}{(r+1)^2}x^2 + \frac{(5+r)^2}{(r+1)^2}x^3 \dots\dots.\right) +$$
$$+x^rC_0\left[-2\frac{(r+2)}{(r+1)^3}x - 2^2\frac{(r+3)}{(r+1)^3}x^2 - 2^3\frac{(r+5)}{(r+1)^3}x^3 \dots\dots..\right]$$

So, the second series solution is

$$y_2 = \left(\frac{\partial y}{\partial r}\right)_{r=0} = C_0(1 + 2^2x + 3^2x^2 + \cdots\dots)logx$$
$$- 2C_0(x + 2.3x^2 + 2^2.5x^3 + \cdots..)$$
$$y_2 = y_1logx - 2C_0(x + 2.3x^2 + 2^2.5x^3 \dots\dots)$$

Now, with $C_0 = 1$, we have the required solution,

$$y = Ay_1 + By_2 = Ay_1 + B[y_1logx - 2(x + 2.3x^2 + 2^2.5x^3 \dots\dots)]$$

$$\Rightarrow y = (A + Blogx)y_1 - 2B(x + 2.3x^2 + 2^2.5x^3 \dots\dots)$$

Thus, we can generalize the procedure as:
If roots of indicial equation equal, i.e. $r = r_1$ (twice), then the two solution of the equation about $x = a$ can be shown to be as:

$$y_1 = (x-a)^{r=r_1}\sum_{n=0}^{\infty}C_n(x-a)^n$$

$$y_2 = y_1\log(x-a) + (x-a)^{r=r_1}\sum_{n=0}^{\infty}\left(\frac{\partial C_n}{\partial r}\right)_{r=r_1}(x-a)^n$$

Ex. 2. Solve the equation in series about $x = 0$

$$(x-x^2)y'' + (1-3x)y' - y = 0$$

Solution:
Here x = 0 is a regular singular point.
Let $y = \sum_{n=0}^{\infty} C_nx^{n+r}, C_0 \neq 0$
be the series solution. Then, we have the indicial equation,

$$[r(r-1) + r]C_0 = 0$$
$$\Rightarrow r^2C_0 = 0$$
$$\Rightarrow r = 0\ (twice)$$

The recurrence relation is,

$$(n+r)(n+r-1)C_n + (n+r)C_n - [(n+r-1)(n+r-2) + 3(n+r-1) + 1]C_{n-1} = 0, n \geq 1$$

$$\Rightarrow (n+r)^2 C_n - (n+r)^2 C_{n-1} = 0, n \geq 1$$

$$\Rightarrow C_n = C_{n-1}, n \geq 1$$

Now for n =1, 2, 3,……

$$C_1 = C_0 = C_2 = C_3 = C_4 \dots \dots \dots$$

Then first series solution for r = 0 is

$$y_1 = C_0(1 + x + x^2 + x^3 + x^4 + \cdots \dots) = \frac{1}{1-x} C_0$$

Now, the second series solution for r =0 is,

$$y_2 = y_1 \log(x) + x^{r=r_1} \sum_{n=0}^{\infty} \left(\frac{\partial C_n}{\partial r}\right)_{r=r_1} x^n = y_1 \log(x) = \frac{logx}{1-x} C_0$$

Thus, with C_0=1, the required solution is

$$y = Ay_1 + By_2 = \frac{(A + Blogx)}{1-x}$$

Assignment:

1. Solve $x^2y'' + 3xy' + (1-x)y = 0$ about $x = 0$.
   Hint.: r = -1 (twice).

$$y_1 = \frac{1}{x} \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

\

$$y_2 = y_1 logx - \frac{2}{x}\left(1 + \sum_{n=1}^{\infty} \frac{H_n x^n}{(n!)^2}\right)$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \dots . . + \frac{1}{n}$