Class II

Series Solution of ODE about a singular point

Ex. 3. Solve the Bessel's equation of order *n*

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

in series about *x* = 0, taking 2*n* as non-integer.

Solution:

Here x = 0 is a regular singularity point. Because,

$$\lim_{x\to 0}(x-0)\left(\frac{x}{x^2}\right) = a\ finite\ quantity(?)$$

$$\lim_{x\to 0}(x-0)^2\left(\frac{x^2-n^2}{x^2}\right) = a\ finite\ quantity(?)$$

Let

$$y = \sum_{m=0}^{\infty} C_m x^{m+r}, C_0 \neq 0$$

be the series solution. Then substituting this series in the differential equation, we get

$$x^2\sum_{m=0}^{\infty}(m+r)(m+r-1)C_m x^{n+r-2} + x\sum_{m=0}^{\infty}(m+r)C_m x^{m+r-1} + x^2\sum_{m=0}^{\infty}C_m x^{m+r} - n^2\sum_{m=0}^{\infty}C_m x^{m+r} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty}(m+r)(m+r-1)C_m x^{m+r} + \sum_{m=0}^{\infty}(m+r)C_m x^{m+r} + \sum_{m=0}^{\infty}C_m x^{m+r+2} - n^2\sum_{m=0}^{\infty}C_m x^{m+r} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty}(m+r)(m+r-1)C_m x^{m+r} + \sum_{m=0}^{\infty}(m+r)C_m x^{m+r}$$
$$+\sum_{p=2}^{\infty} C_{p-2}x^{p+r} - n^2\sum_{m=0}^{\infty} C_m x^{m+r} = 0$$
$$\Rightarrow \sum_{m=0}^{\infty}(m+r)(m+r-1)C_m x^{m+r} + \sum_{m=0}^{\infty}(m+r)C_m x^{m+r} +$$
$$+\sum_{m=2}^{\infty} C_{m-2}x^{m+r} - n^2\sum_{m=0}^{\infty} C_m x^{m+r} = 0$$

The indicial equation is,

$$[r(r-1)+r-n^2]C_0 = 0$$
$$\Rightarrow [r^2-n^2]C_0 = 0$$
$$\Rightarrow r = \pm n$$

So, roots are distinct and assumed that difference of the roots is not an integer.

For n =1, we have

$$(r+1)rC_1 + (r+1)C_1 - n^2C_1 = 0$$

$$\Rightarrow \{(r+1)^2 - n^2\}C_1 = 0$$

For both $r = \pm n$ the factor $\{(r+1)^2 - n^2\} \neq 0$, so $C_1 = 0$.

The recurrence relation is,

$$(m+r)(m+r-1)C_m + (m+r)C_m + C_{m-2} - n^2C_m = 0, m \geq 2$$
$$\Rightarrow (m+r+n)(m+r-n)C_m = -C_{m-2}, m \geq 2$$
$$C_m = -\frac{C_{m-2}}{(m+r+n)(m+r-n)}, m \geq 2$$

which shows that the for all values of m ≥ 2, $C_3 = C_3$ =……= $C_{2m+1} = 0$.

Now, we determine the constants with even suffice.

Now for r = *n*,

$$C_m = -\frac{C_{m-2}}{(m+2n)m}, m \geq 2$$

So, for m = 2, 4, 6,.....2m, we have

$$C_2 = -\frac{C_0}{2^2(n+1)}, \quad C_4 = \frac{C_0}{2^4(2!)(n+1)(n+2)}$$

$$C_6 = -\frac{C_0}{2^6(3!)(n+1)(n+2)(n+3)}, \dots \dots ..$$

$$C_{2m} = (-1)^m \frac{C_0}{2^{2m}(m!)(n+1)(n+2)\dots\dots\dots(n+m)}$$

Thus, the series solution is

$$y = x^r(C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + C_6x^6 \dots \dots)$$

For $r = n$ and taking the corresponding values of constants, first series solution is

$$y_1 = x^n C_0[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^4(2!)(n+1)(n+2)} \cdots + (-1)^m \frac{x^{2m}}{2^{2m}(m!)(n+1)(n+2)\dots.(n+m)}]$$

$$y_1 = C_0[x^n - \frac{x^{n+2}}{2^2(n+1)} + \frac{x^{n+4}}{2^4(2!)(n+1)(n+2)} \cdots + (-1)^m \frac{x^{n+2m}}{2^{2m}(m!)(n+1)(n+2)\dots.(n+m)}]$$

$$y_1 = C_0 \sum_{m=0}^{\infty} (-1)^m \frac{x^{n+2m}}{2^{2m}(m!)(n+1)(n+2)\dots\dots.(n+m)}$$

If we put, $C_0 = \frac{1}{2^n \Gamma(n+1)}$, then

$$y_1 = \sum_{m=0}^{\infty} (-1)^m \frac{x^{n+2m}}{\Gamma(n+1)\, 2^{n+2m} (m!)(n+1)(n+2) \ldots \ldots (n+m)}$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{1}{\Gamma(n+1)\, 2^{n+2m} (m!)(n+1)(n+2) \ldots \ldots (n+m)} \left(\frac{x}{2}\right)^{n+2m}$$

We know that $(n+m)(n+m-1).....(n+1)\,\Gamma(n+1) = \Gamma(n+m+1)$

Therefore,

$$y_1 = \sum_{m=0}^{\infty} (-1)^m \frac{1}{(m!)\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{n+2m} = J_n(x)$$

where $J_n(x)$ is known as the Bessel's function of first kind of order n.

Similiarly, for the other root $r = -n$, we have the second series solution as:

$$y_2 = \sum_{m=0}^{\infty} (-1)^m \frac{1}{(m!)\,\Gamma(-n+m+1)} \left(\frac{x}{2}\right)^{-n+2m} = J_{-n}(x)$$

where $J_{-n}(x)$ is known as the Bessel's function of second kind of order n.

Therefore, the required series solution is:

$$y = AJ_n(x) + BJ_{-n}(x)$$

where A and B are arbitrary constants.