Class I

Series Solution of ODE about a singular point

Case I: Roots of indicial equation are distinct and do not differ by an integer

Ex. 2. Solve in series about $x = 0$

$$2x^2y'' - xy' + (x^2 + 1)y = 0$$

Solution:

Here x = 0 is a regular singularity point. Because,

$$\lim_{x\to 0}(x-0)\left(-\frac{x}{2x^2}\right) = a\ finite\ quantity(?)$$

$$\lim_{x\to 0}(x-0)^2\left(\frac{x^2+1}{2x^2}\right) = a\ finite\ quantity(?)$$

Let

$$y = (x-0)^r\sum_{n=0}^{\infty}(x-0)^n = \sum_{n=0}^{\infty}x^{n+r}, C_0 \neq 0$$

be the series solution. Then substituting this series in the differential equation, we get

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)C_nx^{n+r-2} - x\sum_{n=0}^{\infty}(n+r)C_nx^{n+r-1} + x^2\sum_{n=0}^{\infty}C_nx^{n+r} + \sum_{n=0}^{\infty}C_nx^{n+r} = 0$$

$$\Rightarrow 2\sum_{n=0}^{\infty}(n+r)(n+r-1)C_nx^{n+r} - \sum_{n=0}^{\infty}(n+r)C_nx^{n+r} + \sum_{n=0}^{\infty}C_nx^{n+r+2} + \sum_{n=0}^{\infty}C_nx^{n+r} = 0$$

$$\Rightarrow 2\sum_{n=0}^{\infty}(n+r)(n+r-1)C_nx^{n+r} - \sum_{n=0}^{\infty}(n+r)C_nx^{n+r} +$$

$$+\sum_{m=2}^{\infty} C_{m-2}x^{m+r} + \sum_{n=0}^{\infty} C_n x^{n+r} = 0$$

$$\Rightarrow 2\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)C_n x^{n+r}$$

$$+\sum_{n=2}^{\infty} C_{n-2}x^{n+r} + \sum_{n=0}^{\infty} C_n x^{n+r} = 0$$

The indicial equation is,

$$[2r(r-1) - r + 1]C_0 = 0$$
$$\Rightarrow [(2r-1)(r-1]C_0 = 0$$
$$\Rightarrow r = \frac{1}{2}, 1$$

For n =1, we have

$$2(r+1)rC_1 - (r+1)C_1 + C_1 = 0$$

$$\Rightarrow r(2r+1)C_1 = 0$$

For both r =1, ½ the factor $r(2r+1) \neq 0$, so $C_1 = 0$.

The recurrence relation is,

$$2(n+r)(n+r-1)C_n - (n+r)C_n + C_{n-2} + C_n = 0, n \geq 2$$

$$\Rightarrow (2n+2r-1)(n+r-1)C_n = -C_{n-2}, n \geq 2$$

$$C_n = -\frac{C_{n-2}}{(2n+2r-1)(n+r-1)}, n \geq 2$$

Now for n =2, 3,……

$$C_2 = -\frac{C_0}{(r+1)(2r+3)}, C_3 = -\frac{C_1}{(2r+5)(r+2)} = 0$$

$$C_4 = -\frac{C_2}{(r+3)(2r+7)}$$

$$= -\frac{C_0}{(2r+3)(2r+7)(r+1)(r+3)}, C_5 = 0$$

$$C_6 = -\frac{C_4}{(2r+11)(r+5)}$$

$$= -\frac{C_0}{(2r+3)(2r+7)(2r+11)(r+1)(r+3)(r+5)}$$

Therefore, for r =1,

$$C_2 = -\frac{1}{10}, C_4 = \frac{1}{360}, C_6 = -\frac{1}{78\times 360}, \dots \dots$$

For r = ½,

$$C_2 = -\frac{1}{6}, C_4 = \frac{1}{168}, C_6 = -\frac{1}{66\times168}, \dots \dots$$

The series solution is

$$y = x^r(C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + C_6x^6 \dots\dots)$$

For r= 1 and taking the corresponding values of constants, first series solution is

$$y_1 = x^1C_0\left(1 - \frac{x^2}{10} + \frac{x^4}{360} - \frac{x^6}{78\times360} \dots\dots\right)$$

$$y_1 = xC_0\left(1 - \frac{x^2}{10} + \frac{x^4}{360} - \frac{x^6}{78\times360} \dots\dots\right)$$

with $C_0 = 1$,

$$y_1 = x - \frac{x^3}{10} + \frac{x^5}{360} - \frac{x^7}{78\times360} \dots\dots$$

For r= 1/2 and taking the corresponding values of constants, first series solution is

$$y_2 = x^{1/2}C_0\left(1 - \frac{x^2}{6} + \frac{x^4}{168} - \frac{x^6}{66\times168} \dots\dots\right)$$

with $C_0 = 1$,

$$y_2 = x^{1/2} - \frac{x^{\frac{5}{2}}}{6} + \frac{x^{\frac{9}{2}}}{168} - \frac{x^{\frac{13}{2}}}{78\times360} \dots.$$

Required series solution is

$$y = Ay_1 + By_2$$

Assignment:

1. Solve $2x(1-x)y'' + (1-x)y' + 3y = 0$ in series about $x = 0$