(42)

Numerical Quadrature

unit: 3

06 If the form of a f^n u_x is completely known, then the definite integral $\int_a^b u_x\,dx$ can be evaluated provided its limit exist.

Suppose that the form of the f^n is not completely known but we are given the values of the f^n u_x for a set of equidistant values of the argument x. thus the technique of approximately calculating the values of the integral $\int_a^b u_x\,dx$ is known as numerical integral

06, 00, 02, 03, 04, 05, 08

General quadrature formula for equidistant ordinates

Let $I = \int_a^b y\,dx = \int_a^b f(x)\,dx$

Let, the value of $f(x)$ be given for certain equidistant values of x (say), $x_0, x_0+h, x_0+2h, \ldots, x_0+nh$

Let, the range $[a,b]$ be divided into n equal parts each of length $h = \frac{b-a}{n}$

Let $x_0 = a$, $x_1 = x_0+h = a+h$, $x_2 = a+2h$, $\ldots$ $x_n = a+nh = b$

We have assume that $(n+1)$ ordinates $y_0, y_1, y_2 \ldots y_n$ are equidistant

$$\therefore I = \int_a^b f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx$$

$$= \int_0^n f(x_0+hu)\,h\,du$$

Let, $u = \frac{x-x_0}{h}$

$x - x_0 = hu$

$dx = h\,du$

If $x = x_0$, $u = 0$

$x = x_0+nh$, $u = n$

$$= h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{\lfloor 2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{\lfloor 3}\Delta^3 y_0 + \cdots\right] du$$ (43)

(Applying Newton's forward interpolation formula.)

$$= h\int_0^n \left[y_0 + u\Delta y_0 + \left(\frac{u^2}{2} - \frac{u}{2}\right)\Delta^2 y_0 + \left(\frac{u^3}{6} - \frac{3u^2}{6} + \frac{2u}{6}\right)\Delta^3 y + \cdots\right] du$$

$$= h\left[y_0 u + \frac{u^2}{2}\Delta y_0 + \left(\frac{u^3}{6} - \frac{u^2}{4}\right)\Delta^2 y_0 + \left(\frac{u^4}{24} - \frac{3u^3}{18} + \frac{2u^2}{12}\right)\Delta^3 y_0 + \cdots\right]_0^n$$

$$\therefore \int_{x_0}^{x_0+nh} y\,dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{\lfloor 2} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{\lfloor 3} + \cdots + (n+1) \text{ terms}\right]$$

This is general quadrature formula

The Trapezoidal Rule:

We have the general quadrature formula is

$$\int_{x_0}^{x_0+nh} y\,dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{\lfloor 2} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{\lfloor 3} + \cdots + (n+1) \text{ terms}\right]$$

putting, $n=1$ in the general numerical quadrature formula and neglecting the difference higher than 1

$$\int_{x_0}^{x_0+h} y\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

$$= h\left[\frac{1}{2}(y_0 + y_1)\right]$$

$$= h\left(\frac{y_0 + y_1}{2}\right)$$

|ly,

$$\int_{x_0+h}^{x_0+2h} y\,dx = h\left[\frac{y_1 + y_2}{2}\right]$$

$$\int_{x_0+(n-1)h}^{x_0+nh} y\,dx = h\left[\frac{y_{n-1} + y_n}{2}\right]$$

Now,

$$\int_{x_0}^{x_0+nh} y\,dx = \int_{x_0}^{x_0+h} y\,dx + \int_{x_0+h}^{x_0+2h} y\,dx + \cdots + \int_{x_0+(n-1)h}^{x_0+nh} y\,dx$$

$$= h\left[\frac{y_0+y_1}{2} + \frac{y_1+y_2}{2} + \cdots + \frac{y_{n-1}+y_n}{2}\right]$$

$$= h\left[\frac{y_0+y_n}{2} + (y_1+y_2+\cdots+y_{n-1})\right]$$

which is called the Trapezoidal Rule

0406, **Simpson's $\frac{1}{3}$rd Rule:**

Putting $n=2$ and neglecting the difference higher than two in general quadrature formula, we get

$$\int_{x_0}^{x_0+2h} y\,dx = h\left[2y_0 + 2\Delta y_0 + \left(\frac{8}{3}-\frac{4}{2}\right)\frac{\Delta^2 y_0}{2}\right]$$

$$= h\left[2y_0 + 2\Delta y_0 + \left(\frac{8}{3}-2\right)\frac{\Delta^2 y_0}{2}\right]$$

$$= h\left[\cancel{2y_1 + \tfrac{1}{3}y_2 - \tfrac{2}{3}y_1 + \tfrac{1}{3}y_0}\right]$$

$$= h\left[2y_0 + 2y_1 - 2y_0 + \frac{2}{6}(y_2 - 2y_1 + y_0)\right]$$

$$= h\left[2y_1 + \frac{1}{3}y_2 - \frac{2}{3}y_1 + \frac{1}{3}y_0\right]$$

$$= h\left[\frac{1}{3}y_0 + \frac{4}{3}y_1 + \frac{1}{3}y_2\right]$$

$$= h\left[\frac{1}{3}(y_0+y_2) + \frac{4}{3}y_1\right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

lly,

$$\int_{x_0+2h}^{x_0+4h} y\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$$

$$\int_{x_0+(n-2)h}^{x_0+nh} y\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$$

Now, $\int_{x_0}^{x_0+nh} y\,dx = \int_{x_0}^{x_0+2h} y\,dx + \int_{x_0+2h}^{x_0+4h} y\,dx + \cdots + \int_{x_0+(n-2)h}^{x_0+nh} y\,dx$

$$= \frac{h}{3}\left[y_0+4y_1+y_2+y_2+4y_3+y_4+\cdots+y_{n-2}+4y_{n-1}+y_n\right]$$

$$= \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\cdots+y_{n-1})+2(y_2+y_4+y_6+\cdots+y_{n-2})\right]$$

This eqn is known as Simpson's $\frac{1}{3}$rd rule.

Note In this formula we have neglected all difference above then 2nd so y will be of polynomial of degree 2nd only

$\therefore y = ax^2+bx+c$ (This is also known as parabolic formula)

91, 95, 98, 03, 05 ⊛ Simpson's $3/8$ th rule

putting n = 3 and neglecting the differences higher than the 3rd in general quadrature formula, we get

$$\int_{x_0}^{x_0+3h} y\,dx = h\left[3y_0+\frac{9}{2}\Delta y_0+\left(9-\frac{9}{2}\right)\frac{\Delta^2 y_0}{\lfloor 2}+\left(\frac{81}{4}-27+9\right)\frac{\Delta^3 y_0}{\lfloor 3}\right]$$

$$= h\left[3y_0+\frac{9}{2}(y_1-y_0)+\frac{9}{2}\left(\frac{y_2-2y_1+y_0}{2}\right)+\left(\frac{81-108+36}{4}\right)\left(\frac{y_3-3y_2+3y_1-y_0}{6}\right)\right]$$

$$= h\left[3y_0+\frac{9}{2}(y_1-y_0)+\frac{9}{4}(y_2-2y_1+y_0)+\frac{3}{8}(y_3-3y_2+3y_1-y_0)\right]$$

$$= h\left[3y_0+\frac{9}{2}y_1-\frac{9}{2}y_0+\frac{9}{4}y_2-\frac{9}{2}y_1+\frac{9}{4}y_0+\frac{3}{8}y_3-\frac{9}{8}y_2+\frac{9}{8}y_1-\frac{3}{8}y_0\right]$$

$$= h\left[\left(3-\frac{9}{2}+\frac{9}{4}-\frac{3}{8}\right)y_0+\left(\frac{9}{8}\right)y_1+\left(\frac{9}{4}-\frac{9}{8}\right)y_2+\frac{3}{8}y_3\right]$$

$$= h\left[\frac{3}{8}y_0+\frac{9}{8}y_1+\frac{9}{8}y_2+\frac{3}{8}y_3\right]$$

$$= \frac{3h}{8}(y_0+3y_1+3y_2+y_3)$$

Ily, $\int_{x_0+(n-3)h}^{x_0+nh} y\,dx = \frac{3h}{8}\left[y_{n-3}+3y_{n-2}+3y_{n-1}+y_n\right]$

$$\therefore \int_{x_0}^{x_0+nh} y\,dx = \int_{x_0}^{x_0+3h} y\,dx + \int_{x_0+3h}^{x_0+6h} y\,dx + \cdots + \int_{x_0+(n-3)h}^{x_0+nh} y\,dx$$ (46)

$$= \frac{3h}{8}\Big[y_0+3y_1+3y_2+y_3+y_3+3y_4+3y_5+y_6+\cdots+y_{n-3}+3y_{n-2}+3y_{n-1}+y_n\Big]$$

$$= \frac{3h}{8}\Big[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots+y_{n-1})+2(y_3+y_6+y_9+\cdots+y_{n-3})\Big]$$

Note: Since we have neglected differences above the third so y will be polynomial of third degree.

$\therefore y = ax^3+bx^2+cx+d$

Q9,00,03 Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using Simpson's 1/3 rd and 3/8 th rule. Hence evaluate the approximate value of π in this case.

Sol. We consider the range (0,1) into six equal parts each of width 1/6 and compute the value of $y=\frac{1}{1+x^2}$ at each pt. of subdivision's. The computed values are as follows

	x_0	x_0+h	x_0+2h	x_0+3h	x_0+4h	x_0+5h	x_0+6h
x	0	1/6	2/6	3/6	4/6	5/6	6/6
y_x	1	0.97297	0.90000	0.80000	0.69231	0.59016	0.500000

By Simpson's 1/3 rd rule we have

$$\int_{x_0}^{x_0+6h} y\,dx = \frac{h}{3}\Big[y_0+y_6+4(y_1+y_3+y_5)+2(y_2+y_4)\Big]$$

$$= \frac{1}{18}\Big[(1+0.50000)+4(0.97297+0.80000+0.59016)+2(0.90000+0.69231)\Big]$$

$= \frac{1}{18}[1.5 + 2.45252 + 3.18462]$

$= 0.785397$

By Simpson's 3/8 th rule we get

$$\int_{x_0}^{x_0+6h} y\,dx = \frac{3h}{8}\left[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3\right]$$

$$= \frac{1}{16}[1.5 + 3(0.97297 + 0.90000 + 0.69231 + 0.59016) + 2 \times 0.8000]$$

$$= \frac{1}{16}[1.5 + 9.46632 + 1.6]$$

$$= 0.785395$$

Now $\int_0^1 \frac{dx}{1+x^2} = [\tan^{-1}x]_0^1 = \pi/4$

from 3/8 th rule $\pi/4 = 0.785395 \Rightarrow \pi = 3.14158$

from 1/3 rd rule $\pi/4 = 0.785397 \Rightarrow \pi = 3.141588$

96. Q Use Simpson's 3/8 th rule to obtain an approximate value of $\int_0^{3} (1-2x^3)^{1/3}\,dx$

Sol: Here, $x_0 = 0$, $x_3 = 0.3$ and let $n = 3$ so that $h = 0.1$

Let $y = (1-8x^3)^{1/2}$

$\therefore y_0 = 1$, $y_1 = (1-0.008)^{1/2} = 0.99$, $y_2 = 0.96$, $y_3 = 0.88$

Using Simpson's $\frac{3}{8}$ th rule of four ordinates we get

$$\int_{x_0}^{x_0+3h} y\,dx = \frac{3h}{8}[(y_0+y_3) + 3(y_1+y_2)]$$

$$= \frac{3 \times 0.1}{8}[(1.0+0.89) + 3(0.99+0.97)]$$

$$= \frac{0.3}{8}(1.89 + 5.88)$$

$$= 0.29138$$

x:	x_0 0	x_1 0.1	x_2 0.2	x_3 0.3
y:	1 y_0	0.9959 y_1	0.9674 y_2	0.8854 y_3

Q Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using ① Simpson's $\frac{1}{3}$ rd rule ② Simpson's $\frac{3}{8}$ th rule and compute the result with the exact value $\tan^{-1} 6 = 1.406$ (48)

Soln Let us divide the range (0, 6) into six equal parts, each of width $h=1$ and compute the values of $y = \frac{1}{1+x^2}$ at each pt. of subdivision. The computed values are as follows.

	x_0	x_0+h	x_0+2h	x_0+3h	x_0+4h	x_0+5h	x_0+6h
x:	0	1	2	3	4	5	6
y_x:	1.00	0.500	0.200	0.1000	0.58824	0.038462	0.027027

① using Simpson's $\frac{1}{3}$ rd rule for some ordinates of equal interval with unity, we get:

$$\int_0^6 y\,dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{1}{3}\left[(1+0.027027)+4(0.5+0.1+0.038462)+2(0.2+0.058824)\right]$$

$$= \frac{1}{3}(1.027027+2.553848+0.517648)$$

$$= \frac{1}{3}\times 4.098523 = 1.3661743 = 1.40$$

But, $\int_0^6 \frac{dx}{1+x^2} = \left[\tan^{-1}\right]_0^6 = \tan^{-1}6 - \tan^{-1}0 = 1.40$

Again, Applying Simpson's $\frac{3}{8}$th rule we get

$$\int_0^6 y\,dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right]$$

$$= \frac{3}{8}\left[1.027027+2.391858+0.2\right]$$

$$= \frac{3}{8}\times 3.618885 = 1.357081$$

Q. If the 3rd diff. of $f(x)$ are const. P.T. $\int_{-1}^{1} f(x)\,dx = \frac{2}{3}\left[f(0)+f\left(\frac{1}{\sqrt{2}}\right)+f\left(-\frac{1}{\sqrt{2}}\right)\right]$

Soln The third diff. are given to be const. So we consider $f(x)$ as a polynomial of degree 3.

Let $f(x) = a+bx+cx^2+dx^3$

Now, L.H.S $= \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1}(a+bx+cx^2+dx^3)\,dx$

$= \left[ax+\frac{bx^2}{2}+\frac{cx^3}{3}+\frac{dx^4}{4}\right]_{-1}^{1}$

$= 2(a+c/3)$

and R.H.S $= \frac{2}{3}\left[f(0)+f\left(\frac{1}{\sqrt{2}}\right)+f\left(-\frac{1}{\sqrt{2}}\right)\right]$

$= \frac{2}{3}\left[a+\left(a+\frac{b}{\sqrt{2}}+\frac{c}{2}+\frac{d}{2\sqrt{2}}\right)+\left(a-\frac{b}{\sqrt{2}}+\frac{c}{2}-\frac{d}{2\sqrt{2}}\right)\right]$

$= 2/3\,(3a+c)$

$= 2(a+c/3)$

Q Obtain the approximate formula for $\int_{-1}^{1} u_x\,dx = \frac{13(u_1+u_{-1})-(u_3+u_{-3})}{12}$

Soln since 4 values are given we can set up a polynomial of degree 3

Let, $u_x = a+bx+cx^2+dx^3$

$\therefore \int_{-1}^{1} u_x\,dx = \int_{-1}^{1}(a+bx+cx^2+dx^3)\,dx$

$= \left[ax+\frac{bx^2}{2}+\frac{cx^3}{3}+\frac{dx^4}{4}\right]_{-1}^{1}$

$\Rightarrow \int_{-1}^{1} u_x\,dx = a+b/2+c/3+d/4+a-b/2+c/3-d/4 = 2a+\frac{2c}{3}$

$= \frac{6a+2c}{3}$

$u_1 = a+b+c+d,\; u_{-1} = a-b+c-d$

$\therefore u_3 = a+3b+9c+27d,\; u_{-3} = a-3b+9c-27d$

$\therefore u_1+u_{-1} = 2a+2c = 2(a+c)$

$u_3+u_{-3} = 2a+18c = 2(a+9c)$

$$R.H.S = \frac{13(u_1+u_{-1}) - (u_3+u_{-3})}{12}$$

$$= \frac{13\cdot 2(a+c) - 2(a+9c)}{12}$$

$$= \frac{26a+26c-2a-18c}{12} = \frac{24a+8c}{12} = \frac{6a+2c}{3}$$

$$\therefore \int_{-1}^{1} u_x\,dx = \frac{13(u_1+u_{-1}) - (u_3+u_{-3})}{12}$$

Q S.T if $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ then $\int_0^{2h} f(x)\,dx = \frac{h}{3}\{f(0)+4f(h)+f(2h)\}$. Hence deduce Simpson's rule for numerical integration.

Soln Here, $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

so, $\int_0^{2h} f(x)\,dx = \int_0^{2h} (a_0 + a_1x + a_2x^2 + a_3x^3)\,dx$

$$= \left[a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + a_3\frac{x^4}{4}\right]_0^{2h}$$

$$= 2a_0h + 2a_1h^2 + \frac{8}{3}a_2h^3 + 4a_3h^4]$$

$f(0) = a_0$

$f(h) = a_0 + a_1h + a_2h^2 + a_3h^3$

$f(2h) = a_0 + 2a_1h + 4a_2h^2 + 8a_3h^3$

$\therefore f(0) + 4f(h) + f(2h)$

$\Rightarrow a_0 + 4(a_0 + a_1h + a_2h^2 + a_3h^3) + a_0 + 2a_1h + 4a_2h^2 + 8a_3h^3$

$= 6a_0 + 6a_1h + 8a_2h^2 + 12a_3h^3$

$\Rightarrow \frac{h}{3}[f(0) + 4f(h) + f(2h)] = 2ha_0 + 2a_1h^2 + \frac{8}{3}a_2h^3 + 4a_3h^4$

$\therefore \int_0^{2h} f(x)\,dx = \frac{h}{3}[f(0) + 4f(h) + f(2h)]$

Lly, $\int_{2h}^{4h} f(x)\,dx = \frac{h}{3}[f(2h) + 4f(3h) + f(4h)]$

$\int_{(n-2)h}^{nh} f(x)\,dx = \frac{h}{3}[f(n-2)h + 4f(n-1)h + f(nh)]$

We have

$$\int_0^{nh} f(x)\,dx = \int_0^{2h} f(x)\,dx + \int_{2h}^{4h} f(x)\,dx + \int_{4h}^{6h} f(x)\,dx + \cdots + \int_{(n-2)h}^{nh} f(x)\,dx$$ (51)

$$= \frac{h}{3}\left[f(0) + 4f(h) + f(2h) + f(2h) + 4f(3h) + f(4h) + \cdots + f(n-2)h + f(n-1)h + f(nh)\right]$$

$$= \frac{h}{3}\left[\{f(0) + f(nh)\} + 4\{f(h) + f(3h) + \cdots + f(n-1)h\} + 2\{f(2h) + f(4h) + \cdots + f(n-2)h\}\right]$$

which is Simpson's $\frac{1}{3}$ rd rule.

Q. S.T $\int_0^1 u_x\,dx = \frac{1}{12}(5u_1 + 8u_0 - u_{-1})$

Sol^n Since three variables are given we can get a polynomial of 2nd degree.

Let, $u_x = a + bx + cx^2$

$$\therefore \int_0^1 u_x\,dx = \int_0^1 (a + bx + cx^2)\,dx$$

$$= \left[ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right]_0^1$$

$$= a + b/2 + c/3$$

$\therefore u_1 = a + b + c, \quad u_0 = a, \quad u_{-1} = a - b + c$

$$\therefore 5u_1 + 8u_0 - u_{-1} = 5(a+b+c) + 8a - (a - b + c)$$

$$= 12a + 6b + 4c$$

$$\therefore R.H.S = \frac{1}{12}(5u_1 + 8u_0 - u_{-1})$$

$$= \frac{1}{12}(12a + 6b + 4c)$$

$$= a + b/2 + c/3 = L.H.S$$

$$\therefore \int_0^1 u_x\,dx = \frac{1}{12}(5u_1 + 8u_0 - u_{-1})$$

Imp **Q.** If $f(x)$ is a polynomial of 3rd degree and if $u_{-1} = \int_{-3}^{-1} f(x)\,dx$, $u_0 = \int_{-1}^{1} f(x)\,dx$, $u_1 = \int_1^3 f(x)\,dx$ S.T

$$2f(0) = u_0 - \frac{1}{24}\Delta^2 u_{-1}$$

Sol^n Since $f(x)$ is a polynomial of 3rd degree (52)

Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

$$u_{-1} = \int_{-3}^{-1} f(x)dx = \int_{-3}^{-1} (a_0 + a_1x + a_2x^2 + a_3x^3) dx$$

$$= \left[a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + a_3\frac{x^4}{4}\right]_{-3}^{-1}$$

$$= -a_0 + \frac{a_1}{2} - \frac{a_2}{3} + \frac{a_3}{4} + 3a_0 - \frac{9}{2}a_1 + 9a_2 - \frac{81}{4}a_3$$

$$= 2a_0 - 4a_1 + \frac{26}{3}a_2 - 20a_3$$

$$u_0 = \int_{-1}^{1} f(x)dx = \int_{-1}^{1} (a_0 + a_1x + a_2x^2 + a_3x^3) dx$$

$$= \left[a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + a_3\frac{x^4}{4}\right]_{-1}^{1}$$

$$= 2a_0 + \frac{2a_2}{3}$$

$$u_1 = \int_{1}^{3} f(x)dx = \int_{1}^{3} (a_0 + a_1x + a_2x^2 + a_3x^3) dx$$

$$= \left[a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + a_3\frac{x^4}{4}\right]_{1}^{3}$$

$$= 3a_0 + \frac{9}{2}a_1 + 9a_2 + \frac{81}{4}a_3 - a_0 - \frac{1}{2}a_1 - \frac{1}{3}a_2 - \frac{1}{4}a_3$$

$$\therefore u_1 = 2a_0 + 4a_1 + \frac{26}{3}a_2 + 20a_3$$

Again, $f(0) = a_0$

$$\Rightarrow 2f(0) = 2a_0$$

$$R.H.S = u_0 - \frac{1}{24}\Delta^2 u_{-1}$$

$$= u_0 - \frac{1}{24}(E-1)^2 u_{-1}$$

$$= u_0 - \frac{1}{24}(E^2 - 2E + 1) u_{-1}$$

$$= u_0 - \frac{1}{24}(u_1 - 2u_0 + u_{-1})$$

$$= u_0 - \frac{u_1}{24} + \frac{u_0}{12} - \frac{u_{-1}}{24}$$

$$= \frac{13}{12}u_0 - \frac{u_1}{24} - \frac{u_{-1}}{24}$$

$$= \frac{13}{12}u_0 - \frac{1}{24}(u_1 + u_{-1})$$

$= \frac{13}{12}(2a_0 + \frac{2a_2}{3}) - \frac{1}{24}(2a_0 + 4a_1 + \frac{26}{3}a_2 + 20a_3 + 2a_0 - 4a_1 + \frac{26}{3}a_2 - 20a_3)$

$= \frac{13}{6}a_0 + \frac{13}{18}a_2 - \frac{1}{6}a_0 - \frac{13}{18}a_2$ (53)

$= 2a_0 = 2f(0) = L.H.S$

Q $u_x = a + bx + cx^2$, P.T $\int_1^3 u_x dx = 2u_2 + \frac{1}{12}(u_0 - 2u_2 + u_4)$

Soln Here $u_x = a + bx + cx^2$

$\therefore \int_1^3 u_x dx = \int_1^3 (a + bx + cx^2) dx$

$= \left[ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right]_1^3$

$= 3a + \frac{9}{2}b + 9c - a - \frac{b}{2} - \frac{c}{3}$

$= 2a + 4b + \frac{26}{3}c$

Again, $u_x = a + bx + cx^2$

$u_0 = a$

$u_2 = a + 2b + 4c$

$u_4 = a + 4b + 16c$

$R.H.S = 2u_2 + \frac{1}{12}(u_0 - 2u_2 + u_4)$

$= 2a + 4b + 8c + \frac{1}{12}(a - 2a - 4b - 8c + a + 4b + 16c)$

$= 2a + 4b + 8c + \frac{1}{12}(8c)$

$= 2a + 4b + \frac{26}{3}c = \int_1^3 u_x dx = L.H.S$

2013 Q State Simpson's 3/8 th rule and applying Simpson's 1/3 rd rule to compute the value of π from the formula $\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2}$ by dividing the interval [0,1] into six equal parts give the answer correct in five significant figures.

Soln Statement: The value of the integral $\int_{x_0}^{x_0+nh} f(x) dx$, where $x_n = x_0 + nh$ and n is a multiple of three is given by Simpson's 3/8 th as

$$\int_{x_0}^{x_0+nh} f(x) dx = \frac{3}{8} h \left[(y_0 + y_n) + 3(y_1 + y_2 + \cdots + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3})\right]$$

Here, $\int_0^1 \frac{dx}{1+x^2}$ given $x_0=0$, $x_n=1$, let $n=6$

We have $x_n = x_0 + nh$

$\Rightarrow 1 = 6h \Rightarrow h = 1/6$

x	$y=f(x)$
$x_0=0$	$y_0=1$
$x_1=1/6$	$y_1=0.9729$
$x_2=1/3$	$y_2=0.9000$
$x_3=1/2$	$y_3=0.8000$
$x_4=2/3$	$y_4=0.6923$
$x_5=5/6$	$y_5=0.390$
$x_6=1$	$y_6=0.50$

By Simpson's formula

$$\int_0^1 \frac{dx}{1+x^2} = \frac{1}{3}h\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{1}{18}\left[1+0.50+4(0.9729+0.8+0.390)+2(0.9+0.6923)\right]$$

$$= \frac{1}{18}(1.5+8.6516+3.1846)$$

$$= 0.7409$$

Again, $\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}1 = \pi/4$

$\therefore \pi/4 = 0.7409 \Rightarrow \pi = 2.9636$

Imp Q4. A river is 80 m wide and the depth of the river at a distance x in meters from one of the bank is given by the following table

x:	0	10	20	30	40	50	60	70	80
y:	0	4	7	9	10	15	14	8	3
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Find the approx. area of the cross section.

Soln Area of cross section of the river 80 m wide is given by $\int_0^{80} y_x\,dx$, where, $y_x = d$ = depth of a distance x from one bank.

Here, we are given a equidistant argument at an interval of $h = 10$ (55)

Hence using Simpson's $\frac{1}{3}$rd rule, we get

$$\int_0^{80} y_x\,dx = \frac{1}{3}h\left[(y_0+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)\right]$$

$$= \frac{10}{3}\left[0+3+4(4+9+15+8)+2(7+10+14)\right]$$

$$= \frac{10}{3}\times 209 = 696.66$$

Thus req^d area of cross section is 696.66 sq. metre

Q Applying Simpson's rule find the area betn the ordinate $x=1$ and $x=3$ bounded by the hyperbola $xy=1$

Soln Here, $xy = 1$

$\Rightarrow y = 1/x$

Here, $x_0 = 1$ and $x_n = 3$

Let $n = 6$

$x_n = x_0 + nh$

$3 = 1 + 6h \Rightarrow h = 1/3$

x	$y = f(x)$
$x_0 = 1$	$y_0 = 1$
$x_1 = 4/3$	$y_1 = 3/4$
$x_2 = 5/3$	$y_2 = 3/5$
$x_3 = 2$	$y_3 = 1/2$
$x_4 = 7/3$	$y_4 = 3/7$
$x_5 = 8/3$	$y_5 = 3/8$
$x_6 = 3$	$y_6 = 1/3$

We know from the Simpson's $\frac{1}{3}$rd rule

$$\int_1^3 \frac{1}{x}\,dx = \frac{1}{3}h\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{1}{3}\cdot\frac{1}{3}\left[1+\frac{1}{3}+4\left(\frac{3}{4}+\frac{1}{2}+\frac{3}{8}\right)+2\left(\frac{3}{5}+\frac{3}{7}\right)\right]$$

$$= \frac{1}{9}\left[\frac{4}{3}+4\cdot\frac{13}{8}+2\cdot\frac{36}{35}\right]$$

$$= \frac{1}{9}\left[1.33+6.5+2.05\right] \simeq 1.097$$