n body problem :- Let us consider n bodies with mass ~~m~~ as m_i $(i=1,2,\dots,n)$. Let $\vec{r}_i$ be the position vector of the mass m_i relative to a point O.

Let $\vec{\rho}_{ij}$ be the p.v of the mass m_j relative to the mass m_i.

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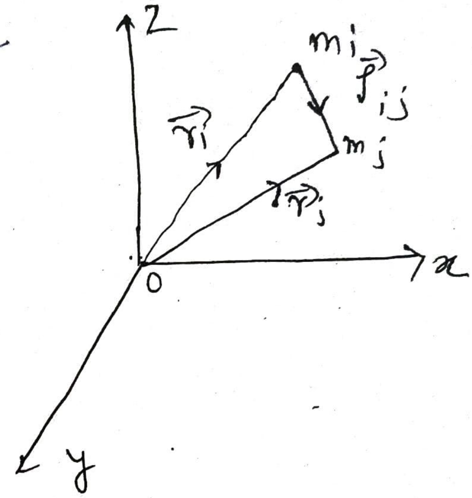

We assume that the masses posses spherical symmetry for which they can be considered as point masses and only ~~extreme~~ external forces are their mutual attraction.

The eq^n of motion of the ith mass is

$$m_i \ddot{\vec{r}}_i = K^2 \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_i m_j}{\rho_{ij}^2} \frac{\vec{\rho}_{ij}}{\rho_{ij}}$$

$$= K^2 \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_i m_j}{\rho_{ij}^3} \vec{\rho}_{ij} \rightarrow (1)$$

summing over i from 1 to n ; we get

$$\sum_{i=1}^{n} m_i \ddot{\vec{r}}_i = K^2 \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_i m_j}{\rho_{ij}^3} \vec{\rho}_{ij} \rightarrow (2)$$

Now, when

$$\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_i m_j}{\rho_{ij}^3} \vec{\rho_{ij}}$$ is expanded we get the terms

of the type $\dfrac{m_i m_j}{\rho_{ij}^3} \vec{\rho_{ij}} + \dfrac{m_j m_i}{\rho_{ij}^3} \vec{\rho_{ji}}$

$$= \frac{m_i m_j}{\rho_{ij}^3} \vec{\rho_{ij}} - \frac{m_i m_j}{\rho_{ij}^3} \vec{\rho_{ij}} \qquad \left[\because \vec{\rho_{ij}} = -\vec{\rho_{ji}}\right]$$

$$= 0$$

(2) $\Rightarrow \sum m_i \ddot{\vec{r_i}} = \vec{0}$

$$\Rightarrow \frac{d^2}{dt^2} \sum m_i \vec{r_i} = \vec{0}.$$

Integrating twice, w.r.t 't' we get

$$\sum m_i \vec{r_i} = \vec{c_1} t + \vec{c_2}$$

where $\vec{c_1}$ & $\vec{c_2}$ are two arbitrary constant vectors

But if r_c is the p.v of the centre of mass of the two masses then

$$\vec{r_c} = \frac{\sum m_i \vec{r_i}}{\sum m_i} = \frac{1}{M} \sum m_i \vec{r_i}$$

$M = \sum m_i$,

$$\Rightarrow M . \vec{r_c} = \vec{c_1} t + \vec{c_2}$$

$$\Rightarrow \vec{r_c} = \frac{\vec{c_1}}{M} t + \frac{\vec{c_2}}{M} \rightarrow (3)$$

From ③, it follows that the locus of the centre of mass is a straight line i.e. the centre of mass is either at rest or in a uniform motion in a straight line.

Eqn ③ is also known as the 1st integral of the n body problem. Again, taking cross product of ① with $\vec{r}_i$, we get

$$\vec{r}_i \times m_i \ddot{\vec{r}}_i = k^2 \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j \, \vec{r}_i \times \vec{\rho}_{ij}}{\rho_{ij}^3}$$

Summing over i from 1 to n, we get

$$\sum_{i=1}^{n} m_i \vec{r}_i \times \ddot{\vec{r}}_i = k^2 \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j \, \vec{r}_i \times \vec{\rho}_{ij}}{\rho_{ij}^3}$$

Now, when the double summation is expanded we get terms of the type

$$\frac{m_i m_j}{\rho_{ij}^3} \vec{r}_i \times \vec{\rho}_{ij} + \frac{m_j m_i}{\rho_{ji}^3} \vec{r}_j \times \vec{\rho}_{ji} = \frac{m_i m_j \, \vec{r}_i \times \vec{\rho}_{ij}}{\rho_{ij}^3} - \frac{m_i m_j \, \vec{r}_j \times \vec{\rho}_{ij}}{\rho_{ij}^3}$$

$$= \frac{m_i m_j}{\rho_{ij}^3} (\vec{r}_i - \vec{r}_j) \times \vec{\rho}_{ij} \quad [\vec{\rho}_{ij} = -\vec{\rho}_{ji}]$$

$$\therefore \frac{m_i m_j}{\rho_{ij}^3}\, \vec{\rho_{ij}} \times \vec{\rho_{ij}} = 0$$

Hence

$$\sum m_i \vec{r_i} \times \ddot{\vec{r_i}} = 0$$

$$\Rightarrow \frac{d}{dt}\left[\sum_{i=1}^{n} \vec{r_i} \times m_i \dot{\vec{r_i}}\right] = 0$$

$$\Rightarrow \sum_{i=1}^{n} \vec{r_i} \times m_i \dot{\vec{r_i}} = \vec{c_3} \quad \rightarrow (4)$$

This shows that the moment of momentum of the system of masses about O or angular momentum of the system about O is constant.

This also know conservation of angular momentum of the masses.

2018 The eq^n (4) is also known as the second integral of the system consisting of n-bodies.

Also, taking dot product of (1) with $\dot{\vec{r_i}}$, we get

$$\dot{\vec{r_i}} \cdot m_i \ddot{\vec{r_i}} = k^2 \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{\rho_{ij}^3}\, \dot{\vec{r_i}} \cdot \vec{\rho_{ij}}$$

Summing i from 1 to n, we get

$$\sum_{i=1}^{n} m_i \dot{\vec{r_i}} \cdot \ddot{\vec{r_i}} = k^2 \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{\rho_{ij}^3}\, \dot{\vec{r_i}} \cdot \vec{\rho_{ij}}$$

The terms on the R.H.S are of the type --

$$\frac{m_i m_j}{\rho_{ij}^3}\vec{r_i}\cdot\vec{\rho_{ij}} + \frac{m_j m_i}{\rho_{ij}^3}\vec{r_j}\cdot\vec{\rho_{ji}}$$

$$= \frac{m_i m_j}{\rho_{ij}^3}(\vec{r_i}-\vec{r_j})\vec{\rho_{ij}}$$

$$= -\frac{m_i m_j}{\rho_{ij}^3}\dot{\vec{\rho_{ij}}}\cdot\vec{\rho_{ij}} = \frac{d}{dt}\left(\frac{m_i m_j}{\rho_{ij}}\right) \quad [\because \vec{\rho_{ij}}\cdot\vec{\rho_{ij}} = \rho_{ij}\dot{\rho}_{ij}]$$

$$\sum_{i=1}^{n} m_i \dot{\vec{r_i}}\,\ddot{\vec{r_i}} = \frac{d}{dt}\left[k^2 \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{m_i m_j}{\rho_{ij}}\right]$$

$$\Rightarrow \frac{d}{dt}\left[\sum_{i=1}^{n}\frac{1}{2}m_i \dot{\vec{r_i}}^2\right] = -\frac{dV}{dt}, \text{ where}$$

$$V = -k^2\sum\sum\frac{m_i m_j}{\rho_{ij}}$$

$$\Rightarrow \frac{dT}{dt}+\frac{dV}{dt} = 0,$$

= Potential energy of the system.

$$\Rightarrow \frac{d}{dt}(T+V) = 0, \quad T = \sum_{i=1}^{n}\frac{1}{2}m_i\dot{\vec{r_i}}^2$$

= K. E of the systen.

$$\Rightarrow T+V = \text{constant} = E\text{ (say)} \rightarrow (5).$$

(5) shows that the total energ. of the systen is constant which is known as the principle of conservation of energy. This is also known as the energy integral of the system of n masses.

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Thus we have deduced three integral (3), (4) & (5) having three constants $\vec{C_1}$, $\vec{C_2}$, $\vec{C_3}$ & one scalar constant E. Thus the soln involve $3 \times 3 + 1 = 10$ arb. const which can not be sufficient to represent the general soln for n body problem. Hence like the three body problem the n-body problem also remain unsolvable.