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De Moivre's Theorem

§ 1. De Moivre's Theorem.

Whatever be the value of n, positive or negative, integral or fractional, the value, or one of the values, of

$$(\cos\theta + i\sin\theta)^n \text{ is } (\cos n\theta + i\sin n\theta).$$

(Meerut 1989, 85S, 85P, 78 ; Jiwaji 80 ; Vikram 78 ; Delhi 81; Kanpur 80 ; Gorakhpur 76 ; Agra 74 ; Alld. 81, 78)

Proof. Case I. Let n be a positive integer.

By actual multiplication, we have

$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$

$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$.

Multiplying both sides by $(\cos\theta_3 + i\sin\theta_3)$, we have

$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)$

$= \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}(\cos\theta_3 + i\sin\theta_3)$

$= \cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)$.

By proceeding in this way, we obtain the product of any number of factors of the form $(\cos\theta + i\sin\theta)$. Thus if there are n factors of such form, we have

$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)......(\cos\theta_n + i\sin\theta_n)$

$= \cos(\theta_1+\theta_2+...+\theta_n) + i\sin(\theta_1+\theta_2+...+\theta_n)$. ...(1)

Putting $\theta_1=\theta_2=...=\theta_n=\theta$ on both sides of (1), we get

$(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)......$ upto n factors

$= \cos(\theta+\theta+...$ upto n terms$) + i\sin(\theta+\theta+...$ upto n terms$)$

$= \cos n\theta + i\sin n\theta$.

i.e., $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

This proves De Moivre's Theorem when n is a positive integer.

Case II. Let n be a negative integer.

Let us suppose $n=-m$, m being a positive integer.

Then $(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$

$= \dfrac{1}{(\cos\theta + i\sin\theta)^m} = \dfrac{1}{(\cos m\theta + i\sin m\theta)}$,

by Case I, since m is a positive integer

$= \dfrac{(\cos m\theta - i\sin m\theta)}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$,

on multiplying the Nr. and Dr. by $(\cos m\theta - i\sin m\theta)$

$= \dfrac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} = \dfrac{\cos m\theta - i\sin m\theta}{1}$

$= \cos(-m\theta) + i\sin(-m\theta)$, $[\because \cos(-m\theta)=\cos m\theta$

and $\sin(-m\theta)=-\sin m\theta]$

$= \cos n\theta + i\sin n\theta$, for $-m=n$.

This proves the De Moivre's Theorem when n is a negative integer.

Remark. If n is a positive integer, then by Case I, we have

$\left(\cos\dfrac{\theta}{n} + i\sin\dfrac{\theta}{n}\right)^n = \cos\left(n.\dfrac{\theta}{n}\right) + i\sin\left(n.\dfrac{\theta}{n}\right)$

$= \cos\theta + i\sin\theta$.

Extracting nth root of both sides, it follows that $\cos(\theta/n) + i\sin(\theta/n)$ is one of the values of $(\cos\theta + i\sin\theta)^{1/n}$.

Thus one of the values of $(\cos\theta + i\sin\theta)^{1/n}$ is obtained by multiplying the angle θ by the index $1/n$.

Case III. Let n be a fraction i.e., rational number, positive or negative.

(Meerut 1985 S)

Suppose $n=p/q$, where q is a positive integer and p is an integer, positive or negative. Then

$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{p/q}$

$= [(\cos\theta + i\sin\theta)^p]^{1/q} = [\cos p\theta + i\sin p\theta]^{1/q}$,

by Case I or II since p is an integer, positive or negative.

Now from the above remark we know that one of the values of $(\cos p\theta + i\sin p\theta)^{1/q}$ must be

$\left(\cos\dfrac{p\theta}{q} + i\sin\dfrac{p\theta}{q}\right)$

i.e., $\cos\dfrac{p\theta}{q} + i\sin\dfrac{p\theta}{q}$ is one of the values of $(\cos\theta + i\sin\theta)^{p/q}$ or of $(\cos\theta + i\sin\theta)^{p/q}$

i.e., $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$, when n is a fraction.

This proves De-Moivre's Theorem when n is a fraction, positive or negative.

Thus De Moivre's Theorem is completely established.

Corollary. *For all values of n, integral or fractional, positive or negative, the value or one of the values of*
$(\cos\theta + i\sin\theta)^n$ *is* $\cos n\theta + i\sin n\theta$.

Proof. We have
$$(\cos\theta - i\sin\theta)^n = [\cos(-\theta) + i\sin(-\theta)]^n$$
$$= \cos(-n\theta) + i\sin(-n\theta), \quad \text{by De Moivre's theorem}$$
$$= \cos n\theta - i\sin n\theta.$$

Note. The students should note the following very carefully :

(i) $(\sin\theta + i\cos\theta)^n \neq \sin n\theta + i\cos n\theta.$

But $(\sin\theta + i\cos\theta)^n = [\cos(\tfrac{1}{2}\pi-\theta) + i\sin(\tfrac{1}{2}\pi-\theta)]^n$
$$= \cos n(\tfrac{1}{2}\pi-\theta) + i\sin n(\tfrac{1}{2}\pi-\theta).$$

(ii) $(\cos\theta + i\sin\phi) \neq \cos n\theta + i\sin n\phi$

i.e., De-Moivre's theorem is applied only when the real and imaginary parts are cosine and sine of the same angle.

(iii) Some authors use the notation cis θ to denote
$$\cos\theta + i\sin\theta.$$

In this notation, De Moivre's theorem would be written as
$$(\text{cis }\theta)^n = \text{cis } n\theta.$$

Results to be remembered.

(i) $(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)\ldots$
$$= \cos(\alpha+\beta+\gamma+\ldots) + i\sin(\alpha+\beta+\gamma+\ldots)$$

i.e., the angles are added.

(ii) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$

(iii) $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta,$

(iv) $(\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta)$
$$= \cos n\theta - i\sin n\theta,$$

(v) $(\cos\theta - i\sin\theta)^{-n} = \cos(-n\theta) - i\sin(-n\theta)$
$$= \cos n\theta + i\sin n\theta,$$

(vi) $\dfrac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta,$

and (vii) $\dfrac{1}{\cos\theta - i\sin\theta} = \cos\theta + i\sin\theta.$

Solved Examples

Ex. 1. *Simplify* $\dfrac{(\cos\theta + i\sin\theta)^4}{(\cos\theta - i\sin\theta)^3}.$

Sol. Since $\cos\theta - i\sin\theta = (\cos\theta + i\sin\theta)^{-1}$, we write

$$\frac{(\cos\theta + i\sin\theta)^4}{(\cos\theta - i\sin\theta)^3} = \frac{(\cos\theta + i\sin\theta)^4}{(\cos\theta + i\sin\theta)^{-3}}$$
$$= (\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta. \quad \text{(Note)}$$

Ex. 2. *Simplify* $\dfrac{(\cos\theta - i\sin\theta)^{10}}{(\cos\theta + i\sin\theta)^{12}}.$

Sol. Since $\cos\theta - i\sin\theta = (\cos\theta + i\sin\theta)^{-1}$, we write
$$\frac{(\cos\theta - i\sin\theta)^{10}}{(\cos\theta + i\sin\theta)^{12}} = \frac{(\cos\theta + i\sin\theta)^{-10}}{(\cos\theta + i\sin\theta)^{12}}$$
$$= (\cos\theta + i\sin\theta)^{-22} = \cos(22\theta) - i\sin(22\theta). \quad \text{(Note)}$$

Ex. 3. *Prove that*
$$\frac{(\cos\theta + i\sin\theta)^8 \, (\cos 2\theta + i\sin 2\theta)^9}{(\cos 2\theta + i\sin 2\theta)^9 \, (\cos\theta - i\sin\theta)^8} = \cos 20\theta - i\sin 20\theta.$$

Sol. Here $\dfrac{(\cos\theta + i\sin\theta)^8 \, (\cos 2\theta + i\sin 2\theta)^9}{(\cos 2\theta + i\sin 2\theta)^9 \, (\cos\theta - i\sin\theta)^8}$
$$= \frac{(\cos\theta + i\sin\theta)^8 \, (\cos\theta + i\sin\theta)^{-18}}{(\cos\theta + i\sin\theta)^{18} \, (\cos\theta + i\sin\theta)^{-8}}$$
$$= (\cos\theta + i\sin\theta)^{8-18-18+8} = (\cos\theta + i\sin\theta)^{-20}$$
$$= \cos 20\theta - i\sin 20\theta. \quad \text{(Note)}$$

Ex. 4. *Prove that*
$$\frac{(\cos 2\theta + i\sin 2\theta)^3 \, (\cos 3\theta - i\sin 3\theta)^4}{(\cos 3\theta + i\sin 3\theta)^2 \, (\cos 4\theta + i\sin 4\theta)^{-3}} = 1.$$

Sol. The given expression on the L.H.S. is
$$= \frac{(\cos\theta + i\sin\theta)^6 \, (\cos\theta + i\sin\theta)^{-12}}{(\cos\theta + i\sin\theta)^6 \, (\cos\theta + i\sin\theta)^{-12}}$$
$$= (\cos\theta + i\sin\theta)^{6-12-6+12} = (\cos\theta + i\sin\theta)^0 = 1. \quad \text{(Note)}$$

*Ex. 5. *Simplify* $\dfrac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^5}.$ **(Meerut 1985 P)**

Sol. We have
$$\frac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^5} = \frac{(\cos\theta + i\sin\theta)^4}{\{\cos(\tfrac{1}{2}\pi-\theta) + i\sin(\tfrac{1}{2}\pi-\theta)\}^5}$$
$$= (\cos\theta + i\sin\theta)^4 \{\cos(\tfrac{1}{2}\pi-\theta) + i\sin(\tfrac{1}{2}\pi-\theta)\}^{-5}$$
$$= (\cos 4\theta + i\sin 4\theta)[\cos\{-(\tfrac{5}{2}\pi - 5\theta)\} + i\sin\{-(\tfrac{5}{2}\pi - 5\theta)\}]$$
$$[\because (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$
$$= \cos(4\theta - \tfrac{5}{2}\pi + 5\theta) + i\sin(4\theta - \tfrac{5}{2}\pi + 5\theta)$$
$$= \cos(9\theta - \tfrac{5}{2}\pi) + i\sin(9\theta - \tfrac{5}{2}\pi)$$
$$= \cos(\tfrac{5}{2}\pi - 9\theta) - i\sin(\tfrac{5}{2}\pi - 9\theta)$$
$$[\because \sin(-\theta) = -\sin\theta \text{ and } \cos(-\theta) = \cos\theta].$$

$= \cos(2\pi + \tfrac{1}{2}\pi - 9\theta) - i\sin(2\pi + \tfrac{1}{2}\pi - 9\theta)$
$= \cos(\tfrac{1}{2}\pi - 9\theta) - i\sin(\tfrac{1}{2}\pi - 9\theta) = \sin 9\theta - i\cos 9\theta.$

Ex. 6. Prove that
$$\left(\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4 = \cos 8\theta + i\sin 8\theta.$$

Sol. We have $\left(\dfrac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4 = \cos 8\theta + i\sin 8\theta.$

$= \dfrac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^4} = \dfrac{(\cos\theta + i\sin\theta)^4}{\{i(\cos\theta + (1/i)\sin\theta)\}^4}$

$= \dfrac{(\cos\theta + i\sin\theta)^4}{i^4[\cos\theta + (1/i)\sin\theta]^4} = \dfrac{(\cos\theta + i\sin\theta)^4}{[\cos\theta - i\sin\theta]^4}$

$\quad [\because\ i^4 = 1,\ 1/i = i/i^2 = i/(-1) = -i]$

$= \dfrac{(\cos\theta + i\sin\theta)^4}{(\cos\theta - i\sin\theta)^4} = \dfrac{(\cos\theta + i\sin\theta)^4}{(\cos\theta + i\sin\theta)^{-4}}$

$= (\cos\theta + i\sin\theta)^8 = \cos 8\theta + i\sin 8\theta.$

Remark. The Ex. 5 can also be solved by the method applied in solving the Ex. 6.

*Ex. 7. If $a = \cos 2\alpha + i\sin 2\alpha$, $b = \cos 2\beta + i\sin 2\beta$, prove that

(i) $a + b = 2\cos(\alpha-\beta)\{\cos(\alpha+\beta) + i\sin(\alpha+\beta)\}$,

(ii) $a - b = 2i\sin(\alpha-\beta)\{\cos(\alpha+\beta) + i\sin(\alpha+\beta)\}$,

and (iii) $\dfrac{a-b}{a+b} = i\tan(\alpha-\beta).$ **[Kanpur 80]**

Sol. (i) We have $a+b = (\cos 2\alpha + i\sin 2\alpha) + (\cos 2\beta + i\sin 2\beta)$
$= (\cos 2\alpha + \cos 2\beta) + i(\sin 2\alpha + \sin 2\beta)$
$= 2\cos(\alpha+\beta)\cos(\alpha-\beta) + 2i\sin(\alpha+\beta)\cos(\alpha-\beta)$
$= 2\cos(\alpha-\beta)[\cos(\alpha+\beta) + i\sin(\alpha+\beta)].$

(ii) Also $a-b = (\cos 2\alpha + i\sin 2\alpha) - (\cos 2\beta + i\sin 2\beta)$
$= (\cos 2\alpha - \cos 2\beta) + i(\sin 2\alpha - \sin 2\beta)$
$= 2\sin(\alpha+\beta)\sin(\beta-\alpha) + 2i\cos(\alpha+\beta)\sin(\alpha-\beta)$
$= 2\sin(\alpha-\beta)[i\cos(\alpha+\beta) - \sin(\alpha+\beta)]$
$= 2\sin(\alpha-\beta)[i\cos(\alpha+\beta) + i^2\sin(\alpha+\beta)]$
$= 2i\sin(\alpha-\beta)[\cos(\alpha+\beta) + i\sin(\alpha+\beta)].\quad [\because\ i^2 = -1]$

(iii) Dividing (ii) by (i), we get
$\dfrac{a-b}{a+b} = \dfrac{2i\sin(\alpha-\beta)[\cos(\alpha+\beta) + i\sin(\alpha+\beta)]}{2\cos(\alpha-\beta)[\cos(\alpha+\beta) + i\sin(\alpha+\beta)]} = i\tan(\alpha-\beta).$

Ex. 8. If $a = \cos 2\alpha + i\sin 2\alpha$, with similar expressions for b, c and d, then prove that

(i) $ab+cd$
$= 2\cos(\alpha+\beta-\gamma-\delta)[\cos(\alpha+\beta+\gamma+\delta) + i\sin(\alpha+\beta+\gamma+\delta)]$

(ii) $(a+b)(c+d)$
$= 4\cos(\alpha-\beta)\cos(\gamma-\delta)[\cos(\alpha+\beta+\gamma+\delta) + i\sin(\alpha+\beta+\gamma+\delta)].$

*(iii) $\sqrt{(abcd)} + \dfrac{1}{\sqrt{(abcd)}} = 2\cos(\alpha+\beta+\gamma+\delta)$ **(Agra 1930)**

(iv) $\dfrac{(a+b)(b+c)(c+a)}{abc}$ is real

(v) $\sqrt{\left\{\dfrac{ab}{cd}\right\}} + \sqrt{\left\{\dfrac{cd}{ab}\right\}} = 2\cos(\alpha+\beta-\gamma-\delta)$

*(vi) $\sqrt{(a^p b^q c^r)} + \dfrac{1}{\sqrt{(a^p b^q c^r)}} = 2\cos(p\alpha+q\beta+r\gamma).$

Sol. (i) We have
$ab = (\cos 2\alpha + i\sin 2\alpha)(\cos 2\beta + i\sin 2\beta)$
$= \cos(2\alpha+2\beta) + i\sin(2\alpha+2\beta)$
$= \cos 2(\alpha+\beta) + i\sin 2(\alpha+\beta)$
$cd = (\cos 2\gamma + i\sin 2\gamma)(\cos 2\delta + i\sin 2\delta)$
$= \cos 2(\gamma+\delta) + i\sin 2(\gamma+\delta)$

$\therefore\ ab+cd = [\cos 2(\alpha+\beta) + \cos 2(\gamma+\delta)]$
$\qquad + i[\sin 2(\alpha+\beta) + \sin 2(\gamma+\delta)]$
$= 2\cos(\alpha+\beta+\gamma+\delta)\cos(\alpha+\beta-\gamma-\delta)$
$\qquad + 2i\sin(\alpha+\beta+\gamma+\delta)\cos(\alpha+\beta-\gamma-\delta)$
$= 2\cos(\alpha+\beta-\gamma-\delta)[\cos(\alpha+\beta+\gamma+\delta)$
$\qquad + i\sin(\alpha+\beta+\gamma+\delta)].$

(ii) Proceeding as in the last Example 7, obtain
$a+b = 2\cos(\alpha-\beta)[\cos(\alpha+\beta) + i\sin(\alpha+\beta)]$
$c+d = 2\cos(\gamma-\delta)[\cos(\gamma+\delta) + i\sin(\gamma+\delta)].$
$\therefore\ (a+b)(c+d) = 4\cos(\alpha-\beta)\cos(\gamma-\delta)$
$\qquad \times[\cos(\alpha+\beta) + i\sin(\alpha+\beta)][\cos(\gamma+\delta) + i\sin(\gamma+\delta)]$
$= 4\cos(\alpha-\beta)\cos(\gamma-\delta)[\cos(\alpha+\beta+\gamma+\delta) + i\sin(\alpha+\beta+\gamma+\delta)].$

(iii) We have
$abcd = (\cos 2\alpha + i\sin 2\alpha)(\cos 2\beta + i\sin 2\beta)(\cos 2\gamma + i\sin 2\gamma)(\cos 2\delta + i\sin 2\delta)$
$= \cos 2(\alpha+\beta+\gamma+\delta) + i\sin 2(\alpha+\beta+\gamma+\delta).$
$\therefore\ \sqrt{(abcd)} = [\cos 2(\alpha+\beta+\gamma+\delta) + i\sin 2(\alpha+\beta+\gamma+\delta)]^{1/2}$
$= \cos(\alpha+\beta+\gamma+\delta) + i\sin(\alpha+\beta+\gamma+\delta).$
and $\dfrac{1}{\sqrt{(abcd)}} = [\cos 2(\alpha+\beta+\gamma+\delta) + i\sin 2(\alpha+\beta+\gamma+\delta)]^{-1/2}$
$= \cos(\alpha+\beta+\gamma+\delta) - i\sin(\alpha+\beta+\gamma+\delta).$ **(Note)**

Adding these, we get
$\sqrt{(abcd)} + \dfrac{1}{\sqrt{(abcd)}} = 2\cos(\alpha+\beta+\gamma+\delta).$

(iv) Proceeding as in Ex. 7, we have

$(a+b) = 2 \cos (\alpha-\beta) [\cos (\alpha+\beta) + i \sin (\alpha+\beta)]$

$(b+c) = 2 \cos (\beta-\gamma) [\cos (\beta+\gamma) + i \sin (\beta+\gamma)]$

$(c+a) = 2 \cos (\gamma-\alpha) [\cos (\gamma+\alpha) + i \sin (\gamma+\alpha)].$

$\therefore (a+b)(b+c)(c+a) = 8 \cos (\alpha-\beta) \cos (\beta-\gamma) \cos (\gamma-\alpha)$
$$\times [\cos (2\alpha+2\beta+2\gamma) + i \sin (2\alpha+2\beta+2\gamma)] \qquad \ldots(1)$$

Also $abc = (\cos 2\alpha + i \sin 2\alpha)(\cos 2\beta + i \sin 2\beta)(\cos 2\gamma + i \sin 2\gamma)$
$$= \cos (2\alpha+2\beta+2\gamma) + i \sin (2\alpha+2\beta+2\gamma). \qquad \ldots(2)$$

Dividing (1) by (2), we get
$$\frac{(a+b)(b+c)(c+a)}{abc} = 8 \cos (\alpha-\beta) \cos (\beta-\gamma) \cos (\gamma-\alpha),$$
which is real.

(v) We have
$$\frac{ab}{cd} = \frac{(\cos 2\alpha + i \sin 2\alpha)(\cos 2\beta + i \sin 2\beta)}{(\cos 2\gamma + i \sin 2\gamma)(\cos 2\delta + i \sin 2\delta)}$$
$$= \frac{\cos (2\alpha+2\beta) + i \sin (2\alpha+2\beta)}{\cos (2\delta+2\delta) + i \sin (2\delta)}$$
$$= \cos 2 (\alpha+\beta-\gamma-\delta) + i \sin 2 (\alpha+\beta-\gamma-\delta).$$

$$\therefore \sqrt{\left(\frac{ab}{cd}\right)} = \cos (\alpha+\beta-\gamma-\delta) + i \sin (\alpha+\beta-\gamma-\delta). \qquad \ldots(1)$$

Also $\sqrt{\left(\frac{cd}{ab}\right)} = \left[\sqrt{\left(\frac{ab}{cd}\right)}\right]^{-1}$, from (1)
$$= [\cos (\alpha+\beta-\gamma-\delta) + i \sin (\alpha+\beta-\gamma-\delta)]^{-1},$$
$$= \cos (\alpha+\beta-\gamma-\delta) + i \sin \{-(2\gamma+2\delta)\}]$$
$$= \cos (\alpha+\beta-\gamma-\delta) - i \sin (\alpha+\beta-\gamma-\delta). \qquad \ldots(2)$$

Adding (1) and (2), we get
$$\sqrt{\left(\frac{ab}{cd}\right)} + \sqrt{\left(\frac{cd}{ab}\right)} = 2 \cos (\alpha+\beta-\gamma-\delta).$$

(vi) We have $a^p b^q c^r$
$$= (\cos 2\alpha + i \sin 2\alpha)^p (\cos 2\beta + i \sin 2\beta)^q (\cos 2\gamma + i \sin 2\gamma)^r$$
$$= \cos 2 (p\alpha + q\beta + r\gamma) + i \sin 2 (p\alpha + q\beta + r\gamma).$$
$$\therefore \sqrt{(a^p b^q c^r)} = [\cos 2 (p\alpha+q\beta+r\gamma) + i \sin 2 (p\alpha+q\beta+r\gamma)]^{1/2}.$$
$$= \cos (p\alpha + q\beta + r\gamma) + i \sin (p\alpha + q\beta + r\gamma), \qquad \ldots(1)$$

and $1/\sqrt{(a^p b^q c^r)} = [\cos 2 (p\alpha+q\beta+r\gamma) + i \sin 2 (p\alpha+q\beta+r\gamma)]^{-1/2}$
$$= \cos (p\alpha+q\beta+r\gamma) - i \sin (p\alpha+q\beta+r\gamma). \qquad \ldots(2)$$

Adding (1) and (2), we get
$$\sqrt{(a^p b^q c^r)} + 1/\sqrt{(a^p b^q c^r)} = 2 \cos (p\alpha+q\beta+r\gamma).$$

Ex. 9. If $x + (1/x) = 2 \cos \theta$, show that $x^n + (1/x^n) = 2 \cos n\theta$.
(Meerut 1988; Bundelkhand 79; Delhi 80)

Sol. Given $x + (1/x) = 2 \cos \theta$.

$\therefore x^2 + 1 = 2x \cos \theta$, or $x^2 - 2x \cos \theta + 1 = 0$.

This gives $x = \dfrac{2 \cos \theta \pm \sqrt{(4 \cos^2 \theta - 4)}}{2} = \cos \theta \pm \sqrt{(\cos^2 \theta - 1)}$

$= \cos \theta \pm \sqrt{(-\sin^2 \theta)} = \cos \theta \pm \sin \theta . \sqrt{(-1)} = \cos \theta \pm i \sin \theta.$

Taking $x = \cos \theta + i \sin \theta$, we have

$x^n + x^{-n} = (\cos \theta + i \sin \theta)^n + (\cos \theta + i \sin \theta)^{-n}$

$= (\cos n\theta + i \sin n\theta) + (\cos n\theta - i \sin n\theta) = 2 \cos n\theta.$

Similarly taking $x = \cos \theta - i \sin \theta = (\cos \theta + i \sin \theta)^{-1}$,

$x^n + (1/x^n) = \cos n\theta + i \sin n\theta + (\cos n\theta - i \sin n\theta) = 2 \cos n\theta.$

*Ex. 10. If $x + (1/x) = 2 \cos \theta$ and $y + (1/y) = 2 \cos \phi$, prove that

(i) One of the values of $x^m y^n + (1/x^m y^n)$ is $2 \cos (m\theta+n\phi)$.
(Vikram 1978; Jiwaji 76, 74; Kanpur 80)

(ii) One of the values of $x^m/y^n + (1/x^m/y^n)$ is $2 \cos (m\theta-n\phi)$.
(Meerut 1987, 84, S-79; Jiwaji 73)

Sol. Since $x + 1/x = 2 \cos \theta$, therefore proceeding as in Ex. 9, we have $x = \cos \theta \pm i \sin \theta.$

$\therefore$ one of the values of x is $\cos \theta + i \sin \theta$.

Similarly one of the values of y is $\cos \phi + i \sin \phi$.

(i) Now taking $x = \cos \theta + i \sin \theta$ and $y = \cos \phi + i \sin \phi$,
we have

$x^m y^n = (\cos \theta + i \sin \theta)^m (\cos \phi + i \sin \phi)^n$

$= (\cos m\theta + i \sin m\theta)(\cos n\phi + i \sin n\phi)$

$= \cos (m\theta + n\phi) + i \sin (m\theta + n\phi).$

Also $\dfrac{1}{x^m y^n} = (x^m y^n)^{-1} = [\cos (m\theta+n\phi) + i \sin (m\theta+n\phi)]^{-1}$

$= \cos (m\theta+n\phi) - i \sin (m\theta+n\phi). \qquad \ldots(2)$

Adding (1) and (2), we have one of the values of

(ii) Taking $x = \cos \theta + i \sin \theta$ and $y = \cos \phi + i \sin \phi$,
we have

$$\frac{x^m}{y^m}=\left(\frac{x}{y}\right)^m=\frac{(\cos\theta+i\sin\theta)^m}{(\cos\phi+i\sin\phi)^m}=\frac{\cos m\theta+i\sin m\theta}{\cos m\phi+i\sin m\phi}$$
$$=(\cos m\theta+i\sin m\theta)(\cos m\phi+i\sin m\phi)^{-1}$$
$$=(\cos m\theta+i\sin m\theta)[\cos(-m\phi)+i\sin(-m\phi)]$$
$$=\cos(m\theta-m\phi)+i\sin(m\theta-m\phi).$$

Also
$$\frac{y^m}{x^m}=\left(\frac{x^m}{y^m}\right)^{-1}=[\cos(m\theta-m\phi)+i\sin(m\theta-m\phi)]^{-1}$$
$$=\cos(m\theta-m\phi)-i\sin(m\theta-m\phi).\qquad\ldots(3)$$

Adding (3) and (4), we have one of the values of

$$\frac{x^m}{y^m}+\frac{y^m}{x^m}\text{ is }2\cos(m\theta-m\phi).$$

Note. If $m=n=1$, this relation becomes
$$\frac{x}{y}+\frac{y}{x}=2\cos(\theta-\phi).$$

Ex. 11. If $x+\dfrac{1}{x}=2\cos\theta$, $y+\dfrac{1}{y}=2\cos\phi$, etc., prove that (Rohilkhand 1976)

(i) $xyz\ldots+\dfrac{1}{xyz\ldots}=2\cos(\theta+\phi+\ldots)$. (Bhopal 1976)

(ii) $x^l y^m z^n\ldots+\dfrac{1}{x^l y^m z^n\ldots}=2\cos(l\theta+m\phi+n\psi+\ldots)$. (Kanpur 1973)

Sol. Since $x+(1/x)=2\cos\theta$,
$x=\cos\theta+i\sin\theta$,
therefore $y=\cos\phi+i\sin\phi$, and so on.
Similarly

(i) We have $xyz\ldots=\cos(\theta+\phi+\ldots)+i\sin(\theta+\phi+\ldots)$ [See Ex. 9]
and $(1/xyz\ldots)=(xyz\ldots)^{-1}=\cos(\theta+\phi+\ldots)-i\sin(\theta+\phi+\ldots).$

Adding, $(xyz\ldots)+(1/xyz\ldots)=2\cos(\theta+\phi+\ldots).$

(ii) We have $x^l=(\cos\theta+i\sin\theta)^l=\cos l\theta+i\sin l\theta$,
$y^m=(\cos\phi+i\sin\phi)^m=\cos m\phi+i\sin m\phi$, and so on.
and $\dfrac{1}{x^l y^m z^n\ldots}=\cos(l\theta+m\phi+\ldots)-i\sin(l\theta+m\phi+\ldots).$

Adding these, we get
$(x^l y^m z^n\ldots)+(1/x^l y^m z^n\ldots)=2\cos(l\theta+m\phi+\ldots).$

Ex. 12. Show that
$$[(\cos\theta+\cos\phi)+i(\sin\theta+\sin\phi)]^n$$
$$+[(\cos\theta+\cos\phi)-i(\sin\theta+\sin\phi)]^n$$
$$=2^{n+1}\cos^n\tfrac{1}{2}(\theta-\phi)\cos^n\tfrac{1}{2}(\theta+\phi).$$

Sol. We have $(\cos\theta+\cos\phi)+i(\sin\theta+\sin\phi)$
$$=2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}+i\,2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}$$

$$=2\cos\frac{\theta-\phi}{2}\left[\cos\frac{\theta+\phi}{2}+i\sin\frac{\theta+\phi}{2}\right].$$

$$\therefore\ [(\cos\theta+\cos\phi)+i(\sin\theta+\sin\phi)]^n$$
$$=2^n\cos^n\frac{\theta-\phi}{2}\left[\cos\frac{\theta+\phi}{2}+i\sin\frac{\theta+\phi}{2}\right]^n$$
$$=2^n\cos^n\frac{\theta-\phi}{2}\left[\cos\frac{n(\theta+\phi)}{2}+i\sin\frac{n(\theta+\phi)}{2}\right].\qquad\ldots(1)$$

Similarly $[(\cos\theta+\cos\phi)-i(\sin\theta+\sin\phi)]^n$
$$=2^n\cos^n\frac{\theta-\phi}{2}\left[\cos\frac{n(\theta+\phi)}{2}-i\sin\frac{n(\theta+\phi)}{2}\right].\qquad\ldots(2)$$

Adding (1) and (2), we get the L.H.S.
$$=2^n\cos^n\frac{\theta-\phi}{2}\cdot2\cos\frac{n(\theta+\phi)}{2}=2^{n+1}\cos^n\frac{\theta-\phi}{2}\cos\frac{n(\theta+\phi)}{2}.$$
$$=\text{the R.H.S.}$$

Ex. 13. Simplify
$$[\{(\cos\theta-\cos\phi)+i(\sin\theta-\sin\phi)\}^n$$
$$+\{(\cos\theta-\cos\phi)-i(\sin\theta-\sin\phi)\}^n].\qquad\text{(Kashmir 1975)}$$

Sol. Proceeding exactly as in Ex. 12, the given expression
$$=2^{n+1}\sin^n\tfrac{1}{2}(\theta-\phi)\cos\tfrac{1}{2}n(\pi+\theta+\phi).$$

Ex. 14. Show that
$$(1+\cos\theta+i\sin\theta)^n+(1+\cos\theta-i\sin\theta)^n$$
$$=2^{n+1}\cos^n(\theta/2)\cos(n\theta/2).$$
(Gorakhpur 1980; Lucknow 76; Magadh 74)

Sol. The L.H.S.
$$=[(1+\cos\theta)+i\sin\theta]^n+[(1+\cos\theta)-i\sin\theta]^n$$
$$=[2\cos^2\tfrac{1}{2}\theta+i.2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta]^n$$
$$+[2\cos^2\tfrac{1}{2}\theta-i.2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta]^n$$
$$=2^n\cos^n(\tfrac{1}{2}\theta)[\cos\tfrac{1}{2}\theta+i\sin\tfrac{1}{2}\theta]^n$$
$$+2^n\cos^n(\tfrac{1}{2}\theta)[\cos\tfrac{1}{2}\theta-i\sin\tfrac{1}{2}\theta]^n$$
$$=2^n\cos^n(\tfrac{1}{2}\theta)[(\cos\tfrac{1}{2}n\theta+i\sin\tfrac{1}{2}n\theta)+(\cos\tfrac{1}{2}n\theta-i\sin\tfrac{1}{2}n\theta)]$$
$$=2^n\cos^n(\tfrac{1}{2}\theta).2\cos\tfrac{1}{2}n\theta=2^{n+1}\cos^n\tfrac{1}{2}\theta.\cos\tfrac{1}{2}n\theta=\text{the R.H.S.}$$

Ex. 15. Prove that
$$\left(\frac{1+\cos\phi+i\sin\phi}{1+\cos\phi-i\sin\phi}\right)^n=\cos n\phi+i\sin n\phi.$$

Sol. The L.H.S.
$$=\left(\frac{1+\cos\phi+i\sin\phi}{1+\cos\phi-i\sin\phi}\right)^n$$
$$=\left(\frac{2\cos^2\tfrac{1}{2}\phi+i.2\sin\tfrac{1}{2}\phi\cos\tfrac{1}{2}\phi}{2\cos^2\tfrac{1}{2}\phi-i.2\sin\tfrac{1}{2}\phi\cos\tfrac{1}{2}\phi}\right)^n$$
$$=\left(\frac{\cos\tfrac{1}{2}\phi+i\sin\tfrac{1}{2}\phi}{\cos\tfrac{1}{2}\phi-i\sin\tfrac{1}{2}\phi}\right)^n=\frac{(\cos\tfrac{1}{2}\phi+i\sin\tfrac{1}{2}\phi)^n}{(\cos\tfrac{1}{2}\phi-i\sin\tfrac{1}{2}\phi)^n}$$
$$=(\cos\tfrac{1}{2}\phi+i\sin\tfrac{1}{2}\phi)^{2n}=\cos n\phi+i\sin n\phi.$$

Ex. 16. *Prove that*

$$\frac{(1+\sin\phi+i\cos\phi)^n}{(1+\sin\phi-i\cos\phi)^n}=\cos\left(\frac{n\pi}{2}-n\phi\right)+i\sin\left(\frac{n\pi}{2}-n\phi\right).$$

(Ranchi 1976 ; Gorakhpur 75 ; Delhi 81 ; Kashmir 72 ; Meerut 82 (P) ; Magadh 72)

Sol. We have

$1+\sin\phi+i\cos\phi=1+\cos(\tfrac{1}{2}\pi-\phi)+i\sin(\tfrac{1}{2}\pi-\phi)$

$=2\cos^2\tfrac{1}{2}(\tfrac{1}{2}\pi-\phi)+2i\sin\tfrac{1}{2}(\tfrac{1}{2}\pi-\phi)\cos\tfrac{1}{2}(\tfrac{1}{2}\pi-\phi)$

$=2\cos\tfrac{1}{2}(\tfrac{1}{2}\pi-\phi)[\cos\tfrac{1}{2}(\tfrac{1}{2}\pi-\phi)+i\sin\tfrac{1}{2}(\tfrac{1}{2}\pi-\phi)].$

$\therefore\quad(1+\sin\phi+i\cos\phi)^n$

$=2^n\cos^n(\tfrac{1}{2}\pi-\tfrac{1}{2}\phi)[\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)+i\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)]^n$

Similarly $1+\sin\phi-i\cos\phi$

$=2\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)[\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)-i\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)].$

$\therefore\quad\frac{(1+\sin\phi+i\cos\phi)^n}{(1+\sin\phi-i\cos\phi)^n}$

$=2^n\cos^n(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)\dfrac{[\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)+i\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)]^n}{2^n\cos^n(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)[\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)-i\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)]^n}$

$=\dfrac{[\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)+i\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)]^n}{[\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)-i\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)]^n}$

$=\dfrac{[\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)+i\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)]^n}{[\{\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)+i\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)\}^{-1}]^n}$

$=[\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)+i\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)]^{2n}$

$=[\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)+i\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)]^{2n}$

$=\cos 2n(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)+i\sin 2n\cdot(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi)$

$=\cos(\tfrac{1}{2}n\pi-n\phi)+i\sin(\tfrac{1}{2}n\pi-n\phi).$

Ex. 17. *Prove that*

$$(a+ib)^{m/n}+(a-ib)^{m/n}=2(a^2+b^2)^{m/2n}\cos\left\{\frac{m}{n}\tan^{-1}\frac{b}{a}\right\}.$$

(Bundelkhand 1980; Lucknow 79)

Sol. Let $a+ib=r(\cos\theta+i\sin\theta)$,

$a-ib=r(\cos\theta-i\sin\theta).$

Equating real and imaginary parts, we have

$a=r\cos\theta$ and $b=r\sin\theta.$

These give $r=\sqrt{(a^2+b^2)}$ and $\tan\theta=b/a$ or $\theta=\tan^{-1}(b/a).$

Now $\quad(a+ib)^{m/n}+(a-ib)^{m/n}$

$=r^{m/n}[(\cos\theta+i\sin\theta)^{m/n}+(\cos\theta-i\sin\theta)^{m/n}]$

$=r^{m/n}\left[\left(\cos\dfrac{m}{n}\theta+i\sin\dfrac{m}{n}\theta\right)+\left(\cos\dfrac{m}{n}\theta-i\sin\dfrac{m}{n}\theta\right)\right]$

$=2r^{m/n}\cos(m/n)\theta=2(a^2+b^2)^{m/2n}\cos\{(m/n)\tan^{-1}(b/a)\},$

putting for r and $\theta.$

Ex. 18. *Prove that*

$$(1+i)^n+(1-i)^n=2^{(n/2)+1}\cos(n\pi/4).$$

(Kanpur 1980 ; Allahabad 82; Gorakhpur 79 ; Ranchi 74; Meerut 83 ; Jiwaji 73 ; Bihar 72 ; Delhi 70)

Sol. Let $\quad1+i=r(\cos\theta+i\sin\theta)$,

$1-i=r(\cos\theta-i\sin\theta).$

Equating real and imaginary parts, we have

$1=r\cos\theta,\ 1=r\sin\theta$ so that $r=\sqrt{2}$ and $\tan\theta=1,$ i.e., $\theta=\tfrac{1}{4}\pi.$

$\therefore\quad(1+i)^n=r^n(\cos\theta+i\sin\theta)^n=r^n(\cos n\theta+i\sin n\theta)$

$(1-i)^n=r^n(\cos\theta-i\sin\theta)^n=r^n(\cos n\theta-i\sin n\theta).$

Adding these, we get

$(1+i)^n+(1-i)^n=2r^n\cos n\theta=2.2^{n/2}\cos\tfrac{1}{4}n\pi,$

putting for r and $\theta.$

$=2^{(n/2)+1}\cos\tfrac{1}{4}n\pi.$

Ex. 19. *If n be a positive integer, prove that*

$$(\sqrt{3}+i)^n+(\sqrt{3}-i)^n=2^{n+1}\cos(n\pi/6).$$ (Lucknow 1975)

Sol. Put $\quad r\cos\theta=\sqrt{3}$ and $r\sin\theta=1,$

so that $r^2=4$ i.e., $r=2$ and $\tan\theta=1/\sqrt{3}$ i.e., $\theta=\pi/6.$

Hence $\quad(\sqrt{3}+i)^n+(\sqrt{3}-i)^n$

$=r^n[(\cos\theta+i\sin\theta)^n+(\cos\theta-i\sin\theta)^n]$

$=2r^n\cos n\theta=2.2^n\cos(n\pi/6),$ putting for r and θ

$=2^{n+1}\cos(n\pi/6).$

Ex. 20. *Express $(1+7i)/(2-i)^2$ in the modulus amplitude form and prove that its fourth power in a real negative number.* (Meerut 1985 S, 80)

Sol. For first part of the question see Ex. 15, chapter 2, page 52. Thus if $u=(1+7i)/(2-i)^2$, then on putting in the modulus-amplitude form, we have

$u=\sqrt{2}[\cos 3\pi/4+i\sin 3\pi/4];$

$\therefore\quad u^4=(\sqrt{2})^4[\cos 3\pi/4+i\sin 3\pi/4]^4$

$=4(\cos 3\pi+i\sin 3\pi),$ by De Moivre's theorem

$=4(-1+i0)=-4,$ which is a real negative number.

Ex. 21. *If $x_r=\cos(\pi/2^r)+i\sin(\pi/2^r)$, prove that*

$$x_1x_2x_3\ldots\text{to inf.}=-1.$$

(Agra 1978 ; Lucknow 78 ; Meerut S-79 ; Jiwaji 80 ; Gorakhpur 78 ; Allahabad 81, 76)

Sol. We are given that

$x_r=\cos(\pi/2^r)+i\sin(\pi/2^r),$

Putting $r=1, 2, 3\ldots$ in (1) and multiplying these expressions,

$x_1x_2x_3\ldots.\text{to inf.}$ ...(1)

we get

$=\left(\cos\dfrac{\pi}{2}+i\sin\dfrac{\pi}{2}\right)\left(\cos\dfrac{\pi}{2^2}+i\sin\dfrac{\pi}{2^2}\right)\left(\cos\dfrac{\pi}{2^3}+i\sin\dfrac{\pi}{2^3}\right)\ldots\text{to }\infty$

$=\cos\left(\dfrac{\pi}{2}+\dfrac{\pi}{2^2}+\dfrac{\pi}{2^3}+\ldots\text{to }\infty\right)+i\sin\left(\dfrac{\pi}{2}+\dfrac{\pi}{2^2}+\dfrac{\pi}{2^3}+\ldots\text{to }\infty\right)$

$$= \cos\left[\frac{\pi/2}{1-\frac{\pi}{4}}\right] + i \sin\left[\frac{\pi/2}{1-\frac{\pi}{4}}\right], \text{ summing the G.P.}$$

$$= \cos \pi + i \sin \pi = -1 + i.0 = -1. \qquad \text{Proved.}$$

*Ex. 22. If
$$(a_1+ib_1)(a_2+ib_2)...(a_n+ib_n) = A+iB,$$
prove that
$$(a_1^2+b_1^2)(a_2^2+b_2^2)...(a_n^2+b_n^2) = A^2+B^2$$
and
$$\tan^{-1}\frac{b_1}{a_1} + \tan^{-1}\frac{b_2}{a_2} + ... + \tan^{-1}\frac{b_n}{a_n} = \tan^{-1}\frac{B}{A}.$$

(Banaras 1977 ; Vikram 77 ; Gorakhpur 82, 76)

Sol. Let $a_1+ib_1 = r_1(\cos\theta_1+i\sin\theta_1)$,
$a_2+ib_2 = r_2(\cos\theta_2+i\sin\theta_2),..., a_n+ib_n = r_n(\cos\theta_n+i\sin\theta_n)$.

Then $a_1 = r_1\cos\theta_1,\ b_1 = r_1\sin\theta_1$,

so that $r_1^2 = a_1^2+b_1^2,\ \theta_1 = \tan^{-1}(b_1/a_1)$,

and similar other expressions for $r_2, \theta_2, ..., r_n, \theta_n$.

Also, let $A+iB = R(\cos\phi+i\sin\phi)$,

so that $R^2 = A^2+B^2$ and $\phi = \tan^{-1}(B/A)$.

Now it is given that
$$(a_1+ib_1)(a_2+ib_2)...(a_n+ib_n) = (A+iB).$$

$\therefore [r_1(\cos\theta_1+i\sin\theta_1)]...(a_n+ib_n) = (A+iB)$.

or $r_1r_2...r_n[\cos(\theta_1+\theta_2+...+\theta_n)+i\sin(\theta_1+\theta_2+...+\theta_n)]$

$$= R(\cos\phi+i\sin\phi).$$

Equating real and imaginary parts on both sides of (1), we get

$r_1r_2...r_n\cos(\theta_1+\theta_2+...+\theta_n) = R\cos\phi,$ ...(2)

and $r_1r_2...r_n\sin(\theta_1+\theta_2+...+\theta_n) = R\sin\phi.$ ...(3)

Squaring and adding (2) and (3), we have
$$r_1^2r_2^2...r_n^2 = R^2$$

or $(a_1^2+b_1^2)(a_2^2+b_2^2)...(a_n^2+b_n^2) = A^2+B^2,$

which proves the first result.

Dividing (3) by (2), we have

$\tan\phi = \tan(\theta_1+\theta_2+...+\theta_n)$

or $\phi = \theta_1+\theta_2+...+\theta_n$

or $\tan^{-1}(B/A) = \tan^{-1}(b_1/a_1)+...+\tan^{-1}(b_n/a_n),$

which proves the second result.

Ex. 23. If $\left(1+i\frac{x}{a}\right)\left(1+i\frac{x}{b}\right)\left(1+i\frac{x}{c}\right)... = A+iB$,

prove that
$$\left(1+\frac{x^2}{a^2}\right)\left(1+\frac{x^2}{b^2}\right)\left(1+\frac{x^2}{c^2}\right)... = A^2+B^2,$$
and
$$\tan^{-1}(x/a)+\tan^{-1}(x/b)+\tan^{-1}(x/c)+... = \tan^{-1}(B/A).$$

(Kanpur 1980)

Sol. Let $1+i(x/a) = r_1(\cos\theta_1+i\sin\theta_1)$,
$1+i(x/b) = r_2(\cos\theta_2+i\sin\theta_2)$, and so on.

Then $1 = r_1\cos\theta_1,\ x/a = r_1\sin\theta_1$,

so that $r_1^2 = 1+(x^2/a^2),\ \theta_1 = \tan^{-1}(x/a)$.

Similarly $r_2^2 = 1+(y^2/b^2),\ \theta_2 = \tan^{-1}(y/b)$,

and similar other expressions for r_3, θ_3 and so on.

Now it is given that
$$\left(1+i\frac{x}{a}\right)\left(1+i\frac{x}{b}\right)\left(1+i\frac{x}{c}\right)... = A+iB.$$

$\therefore [\cos(\theta_1+\theta_2+...)+i\sin(\theta_1+\theta_2+...)][r_2(\cos\theta_2+i\sin\theta_2+...)] = A+iB.$

or $r_1r_2r_3...\cos(\theta_1+\theta_2+...)+i\sin(\theta_1+\theta_2+\theta_3+...) = A+iB.$

Equating real and imaginary parts on both sides, we get

$A = r_1r_2r_3...\cos(\theta_1+\theta_2+\theta_3+...),$ ...(1)

and $B = r_1r_2r_3...\sin(\theta_1+\theta_2+\theta_3+...).$ ...(2)

Squaring and adding (1) and (2), we get
$$A^2+B^2 = r_1^2r_2^2r_3^2... = \left(1+\frac{x^2}{a^2}\right)\left(1+\frac{x^2}{b^2}\right)\left(1+\frac{x^2}{c^2}\right)...$$

which proves the first result.

Dividing (2) by (1), we get

$B/A = \tan(\theta_1+\theta_2+\theta_3+...)$

or $\tan^{-1}(B/A) = \theta_1+\theta_2+\theta_3+...$

$= \tan^{-1}(x/a)+\tan^{-1}(x/b)+\tan^{-1}(x/c)+...,$

which proves the second result.

**Ex. 24. If $\cos\alpha+\cos\beta+\cos\gamma = \sin\alpha+\sin\beta+\sin\gamma = 0$,

prove that
$$\cos 3\alpha+\cos 3\beta+\cos 3\gamma = 3\cos(\alpha+\beta+\gamma),$$
and $\sin 3\alpha+\sin 3\beta+\sin 3\gamma = 3\sin(\alpha+\beta+\gamma).$

(Meerut 1988 ; Kanpur 80 ; Agra 78 ; Lucknow 81)

Sol. Let $a = \cos\alpha+i\sin\alpha,\ b = \cos\beta+i\sin\beta,\ c = \cos\gamma+i\sin\gamma.$

Then $a+b+c = (\cos\alpha+\cos\beta+\cos\gamma)+i(\sin\alpha+\sin\beta+\sin\gamma)$

$= 0+i0 = 0.$

[∵ as given, $\cos\alpha+\cos\beta+\cos\gamma = 0 = \sin\alpha+\sin\beta+\sin\gamma$]

Now $a+b+c = 0 \Rightarrow a+b = -c.$

$\Rightarrow (a+b)^3 = -c^3$

$\Rightarrow a^3+b^3+3ab(a+b) = -c^3$

$\Rightarrow a^3+b^3+3ab(-c) = -c^3$

$\Rightarrow a^3+b^3+c^3 = 3abc.$ [∵ $a+b=-c$]   ...(1)

Substituting for a, b, c in (1), we have

$(\cos\alpha+i\sin\alpha)^3+(\cos\beta+i\sin\beta)^3+(\cos\gamma+i\sin\gamma)^3$

$= 3(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma).$

Applying De Moivre's Theorem, we get

$(\cos 3\alpha + i \sin 3\alpha) + (\cos 3\beta + i \sin 3\beta) + (\cos 3\gamma + i \sin 3\gamma)$

$= 3 \{\cos (\alpha+\beta+\gamma) + i \sin (\alpha+\beta+\gamma)\}$

or $(\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i (\sin 3\alpha + \sin 3\beta + \sin 3\gamma)$
$= 3 \cos (\alpha+\beta+\gamma) + i 3 \sin (\alpha+\beta+\gamma)$.

Equating real and imaginary parts on both sides, we get

$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos (\alpha+\beta+\gamma)$.

and $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin (\alpha+\beta+\gamma)$.

Ex. 25. *If* $\cos \alpha + \cos \beta + \cos \gamma = \sin \alpha + \sin \beta + \sin \gamma = 0$,
prove that $\Sigma \cos 4\alpha = 2\Sigma \cos 2 (\beta+\gamma)$ *and* $\Sigma \sin 4\alpha = 2\Sigma \sin 2 (\beta+\gamma)$.

Sol. Let $x = \cos \alpha + i \sin \alpha$, $y = \cos \beta + i \sin \beta$, $z = \cos \gamma + i \sin \gamma$,

∴ $x+y+z = (\cos \alpha + i \sin \alpha) + (\cos \beta + i \sin \beta) + (\cos \gamma + i \sin \gamma)$

$= (\cos \alpha + \cos \beta + \cos \gamma) + i (\sin \alpha + \sin \beta + \sin \gamma)$

$= 0 + i.0 = 0$,   (as given).

Now $x+y+z = 0$ gives $x+y = -z$

$(x+y)^3 = -z^3$ (squaring both sides)

$x^2+y^2-z^2 = -2xy$

$(x^2+y^2-z^2)^2 = 4x^2y^2$,   (again squaring both sides)

$x^4+y^4+z^4-2x^2y^2-2x^2z^2-2y^2z^2 = 2 (x^2y^2+y^2z^2+z^2x^2)$

$x^4+y^4+z^4 = 2 (x^2y^2+y^2z^2+z^2x^2)$

or $\Sigma x^4 = 2\Sigma y^2z^2$, (expressing in the summation notation)

$\Sigma (\cos \alpha + i \sin \alpha)^4 = 2\Sigma (\cos \beta + i \sin \beta)^2 (\cos \gamma + i \sin \gamma)^2$,
putting for x, y and z from (1)

or $\Sigma (\cos 4\alpha + i \sin 4\alpha) = 2\Sigma (\cos 2\beta + i \sin 2\beta) (\cos 2\gamma + i \sin 2\gamma)$
$= 2\Sigma [\cos 2 (\beta+\gamma) + i \sin 2 (\beta+\gamma)]$.

Equating real and imaginary parts on both sides, we get

$\Sigma \cos 4\alpha = 2\Sigma \cos 2 (\beta+\gamma)$

and $\Sigma \sin 4\alpha = 2\Sigma \sin 2 (\beta+\gamma)$.

Ex. 26. *If*
$x = \cos \alpha + i \sin \alpha$, $y = \cos \beta + i \sin \beta$,
$z = \cos \gamma + i \sin \gamma$ and $x+y+z = xyz$,
then prove that
$\cos (\beta-\gamma) + \cos (\gamma-\alpha) + \cos (\alpha-\beta) + 1 = 0$.

Sol. Since $x+y+z = xyz$, therefore
$(\cos \alpha + i \sin \alpha) + (\cos \beta + i \sin \beta) + (\cos \gamma + i \sin \gamma)$
$= (\cos \alpha + i \sin \alpha) (\cos \beta + i \sin \beta) (\cos \gamma + i \sin \gamma)$
$= \cos (\alpha+\beta+\gamma) + i \sin (\alpha+\beta+\gamma)$.

Equating real and imaginary parts, we have
$\cos \alpha + \cos \beta + \cos \gamma = \cos (\alpha+\beta+\gamma)$,
and $\sin \alpha + \sin \beta + \sin \gamma = \sin (\alpha+\beta+\gamma)$.

Squaring and adding these, we get
$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + 2 \cos \beta \cos \gamma + 2 \cos \gamma \cos \alpha + 2 \cos \alpha \cos \beta$
$+ \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma + 2 \sin \beta \sin \gamma + 2 \sin \gamma \sin \alpha + 2 \sin \alpha \sin \beta = 1$.

or $1+1+1 + 2 \cos (\beta-\gamma) + 2 \cos (\gamma-\alpha) + 2 \cos (\alpha-\beta) = 1$

or $\cos (\beta-\gamma) + \cos (\gamma-\alpha) + \cos (\alpha-\beta) + 1 = 0$.

Ex. 27. *If* $x = \cos \alpha + i \sin \alpha$, $y = \cos \beta + i \sin \beta$, $z = \cos \gamma + i \sin \gamma$,
and if $x+y+z = 0$, *then prove that*
$1/x + 1/y + 1/z = 0$.   **(Kanpur P. 1980; Jiwaji 78)**

Sol. Since $x = \cos \alpha + i \sin \alpha$,
$1/x + 1/y + 1/z = x^{-1} + y^{-1} + z^{-1}$
$= (\cos \alpha + i \sin \alpha)^{-1} + (\cos \beta + i \sin \beta)^{-1} + (\cos \gamma + i \sin \gamma)^{-1}$
$= \cos \alpha - i \sin \alpha + \cos \beta - i \sin \beta + \cos \gamma - i \sin \gamma$
$= (\cos \alpha + \cos \beta + \cos \gamma) - i (\sin \alpha + \sin \beta + \sin \gamma)$
$= 0 - i.0 = 0$, from (1).

Ex. 28. *If* $\sin \alpha + \sin \beta + \sin \gamma = \cos \alpha + \cos \beta + \cos \gamma = 0$,
then prove that
(i) $\Sigma \sin 2\alpha = \Sigma \cos^2 \alpha$,
(ii) $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = \cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$.

Sol. Let $x = \cos \alpha + i \sin \alpha$, $y = \cos \beta + i \sin \beta$,
$z = \cos \gamma + i \sin \gamma$.

Then $x+y+z = (\cos \alpha + \cos \beta + \cos \gamma) + i (\sin \alpha + \sin \beta + \sin \gamma)$

$= 0 + i0 = 0$.   ...(1)

Also $1/x + 1/y + 1/z = x^{-1} + y^{-1} + z^{-1}$
$= (\cos \alpha + i \sin \alpha)^{-1} + (\cos \beta + i \sin \beta)^{-1} + (\cos \gamma + i \sin \gamma)^{-1}$
$= (\cos \alpha - i \sin \alpha) + (\cos \beta - i \sin \beta) + (\cos \gamma - i \sin \gamma)$
$= (\cos \alpha + \cos \beta + \cos \gamma) - i (\sin \alpha + \sin \beta + \sin \gamma)$
$= 0 - i0 = 0$.   ...(2)

Now $(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2zx$
$= x^2+y^2+z^2+2xyz (1/x+1/y+1/z)$

or $0 = x^3 + y^3 + z^3 + 2xyz.0,$

       $x^3 + y^3 + z^3 = 0.$

$\therefore$ $(\cos \alpha + i \sin \alpha)^2 + (\cos \beta + i \sin \beta)^2 + (\cos \gamma + i \sin \gamma)^2 = 0$

or $(\cos 2\alpha + i \sin 2\alpha) + (\cos 2\beta + i \sin 2\beta) + (\cos 2\gamma + i \sin 2\gamma) = 0,$

or $(\cos 2\alpha + \cos 2\beta + \cos 2\gamma) + i (\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = 0.$

Equating real and imaginary parts on both sides, we get

       $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0.$

Now $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

$\Rightarrow$ $(\cos^2 \alpha - \sin^2 \alpha) + (\cos^2 \beta - \sin^2 \beta) + (\cos^2 \gamma - \sin^2 \gamma) = 0$

$\Rightarrow$ $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma$

$\Rightarrow$ $\Sigma \cos^2 \alpha = \Sigma \sin^2 \alpha.$

Ex. 29. *Find the general value of θ which satisfies the equation*
$(\cos \theta + i \sin \theta)(\cos 2\theta + i \sin 2\theta)\ldots(\cos n\theta + i \sin n\theta) = 1.$

Sol. The given equation is

$(\cos \theta + i \sin \theta)(\cos 2\theta + i \sin 2\theta)\ldots(\cos n\theta + i \sin n\theta) = 1$

or $\cos (\theta + 2\theta + 3\theta + \ldots + n\theta) + i \sin (\theta + 2\theta + 3\theta + \ldots + n\theta) = 1$

or $\cos (1 + 2 + 3 + \ldots + n) \theta + i \sin (1 + 2 + 3 + \ldots + n) \theta = 1$

or $\cos \left(\dfrac{n(n+1)}{2} \theta \right) + i \sin \left(\dfrac{n(n+1)}{2} \theta \right) = 1.$

$$\left[\because \ 1 + 2 + \ldots + n = \frac{n(n+1)}{2} \right]$$

Equating real and imaginary parts, we have

$\cos \dfrac{n(n+1)}{2} \theta = 1,$ ...(1)

and $\sin \dfrac{n(n+1)}{2} \theta = 0.$ ...(2)

From the equations (1) and (2), we have

$\dfrac{n(n+1)}{2} \theta = 2r\pi,$ where r is any integer

$\therefore$ $\theta = \dfrac{4r\pi}{n(n+1)},$ where r is any integer.

Ex. 30. *Find the general value of θ which satisfies the equation*

$(\cos \theta + i \sin \theta)(\cos 3\theta + i \sin 3\theta)\ldots\{\cos (2r-1) \theta + i \sin (2r-1) \theta\} = 1.$
(Rohilkhand 1979)

Sol. The given equation is

$(\cos \theta + i \sin \theta)(\cos 3\theta + i \sin 3\theta)\ldots\{\cos (2r-1)\theta + i \sin (2r-1)\theta\} = 1$

or $\cos [\theta + 3\theta + \ldots + (2r-1) \theta] + i \sin [\theta + 3\theta + \ldots + (2r-1) \theta] = 1$

or $\cos [1 + 3 + \ldots + (2r-1)] \theta + i \sin [1 + 3 + \ldots + (2r-1)] \theta = 1$

or $\cos r^2\theta + i \sin r^2\theta = 1.$

$$\left[\because \text{the sum of the A.P. } 1 + 3 + \ldots + (2r-1) = \tfrac{1}{2}r \{2 + (r-1) 2\} = r^2 \right]$$

Equating real and imaginary parts, we get

$\cos r^2\theta = 1$ and $\sin r^2\theta = 0.$

Hence the general value of θ is given by

$r^2\theta = 2n\pi,$ or $\theta = (2n\pi/r^2),$ where n is any integer.

Ex. 31. *If α and β are the roots of the equation*

$$x^2 - 2x + 4 = 0,$$

prove that

$$\alpha^n + \beta^n = 2^{n+1} \cos (n\pi/3).$$

(Meerut 1937P, 84S, 83, Agra 79, Jiwaji 78; Kanpur 77; Bundelkhand 76; Rohilkhand 78)

Sol. The roots of the equation $x^2 - 2x + 4 = 0$ are given by

$x = \dfrac{2 \pm \sqrt{(4-16)}}{2} = \dfrac{2 \pm 2\sqrt{3}i}{2} = 1 \pm i\sqrt{3}.$

Therefore let $\alpha = 1 + i\sqrt{3}$ and $\beta = 1 - i\sqrt{3}.$

Now to change α and β to modulus-amplitude form, we put

$1 = r \cos \theta$ and $\sqrt{3} = r \sin \theta.$

$\therefore$ $r^2 = 4$ or $r = 2.$

$\cos \theta = 1/2$ and $\sin \theta = \sqrt{3}/2,$ giving $\theta = \tfrac{1}{3}\pi.$

Then $\alpha = 2 (\cos \tfrac{1}{3}\pi + i \sin \tfrac{1}{3}\pi)$ and $\beta = 2 (\cos \tfrac{1}{3}\pi - i \sin \tfrac{1}{3}\pi).$

Now $\alpha^n + \beta^n = 2^n [(\cos \tfrac{1}{3}\pi + i \sin \tfrac{1}{3}\pi)^n + (\cos \tfrac{1}{3}\pi - i \sin \tfrac{1}{3}\pi)^n]$

$= 2^n [\cos (n.\tfrac{1}{3}\pi) + i \sin (n.\tfrac{1}{3}\pi) + \cos (n.\tfrac{1}{3}\pi) - i \sin (n.\tfrac{1}{3}\pi)]$

$= 2^n.2 \cos \tfrac{1}{3}n\pi = 2^{n+1} \cos \tfrac{1}{3}n\pi.$

Ex. 32. *Find the equation whose roots are the nth powers of the roots of the equation $x^2 - 2x \cos \theta + 1 = 0.$*

(Meerut 1986 S; 85 P, Kanpur 1978)

Sol. The roots of the equation $x^2 - 2x \cos \theta + 1 = 0$ are

$x = \dfrac{2 \cos \theta \pm \sqrt{(4 \cos^2 \theta - 4)}}{2} = \cos \theta \pm \sqrt{(\cos^2 \theta - 1)}$

$= \cos \theta \pm \sqrt{(-\sin^2\theta)} = \cos \theta \pm i \sin \theta.$

Let $\alpha = \cos \theta + i \sin \theta$ and $\beta = \cos \theta - i \sin \theta.$

We are to find the equation whose roots are α^n and $\beta^n.$

We have $\alpha^n = (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta,$

$\beta^n = (\cos \theta - i \sin \theta)^n = \cos n\theta - i \sin n\theta.$

$\therefore$ $\alpha^n + \beta^n = 2 \cos n\theta,$

$\alpha^n\beta^n = (\cos n\theta + i \sin n\theta) (\cos n\theta - i \sin n\theta)$

$= \cos^2 n\theta + \sin^2 n\theta = 1.$

Now the equation whose roots are α^n and β^n is

$x^2 - (\text{sum of the roots}) \ x + \text{product of the roots} = 0$

$$x^2 - (\alpha^n + \beta^n) x + \alpha^n \beta^n = 0$$

i.e.,

or $\qquad x^2 - 2x \cos n\theta + 1 = 0.$

Ex. 33. *Solve* $x^2 - 2ax \cos \theta + a^2 = 0$ *and show that if* α *is either root of this equation, then* $\alpha^{2n} - 2a^n \alpha^n \cos n\theta + a^{2n} = 0$, *where* n *is a positive integer.*

Sol. The roots of the equation $x^2 - 2ax \cos \theta + a^2 = 0$ are

$$x = \frac{2a \cos \theta \pm \sqrt{(4a^2 \cos^2 \theta - 4a^2)}}{2} = a (\cos \theta \pm i \sin \theta).$$

As given, $\quad \alpha = a (\cos \theta + i \sin \theta)$, so that

$\alpha^n = a^n (\cos n\theta + i \sin n\theta)$ and $\quad \alpha^{2n} = a^{2n} (\cos 2n\theta + i \sin 2n\theta).$

$\alpha^{2n} - 2a^n \alpha^n \cos n\theta + a^{2n}$

$= a^{2n} (\cos 2n\theta \pm i \sin 2n\theta) - 2a^n (\cos n\theta \pm i \sin n\theta) + a^{2n}$

$= a^{2n} (\cos 2n\theta \pm i \sin 2n\theta) - 2a^{2n} \cos n\theta (\cos n\theta \pm i \sin n\theta) + a^{2n}$

$= a^{2n} \{\cos 2n\theta \pm i \sin 2n\theta - 2 \cos^2 n\theta \mp 2i \sin n\theta \cos n\theta + 1\}$

$= a^{2n} \{\cos 2n\theta \pm i \sin 2n\theta \mp i \sin 2n\theta - (2 \cos^2 n\theta - 1)\}$

$= a^{2n} \{\cos 2n\theta - \cos 2n\theta\} = 0.$

Ex. 34. *Find the value of* x *such that*

$$\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = \frac{\sin n\theta}{\sin^n \theta},$$

where α *and* β *are the roots of* $t^2 - 2t + 2 = 0$. \qquad (**Lucknow 1981**)

Sol. Solving the equation $t^2 - 2t + 2 = 0$, we have

$$t = \frac{2 \pm \sqrt{(4-8)}}{2} = 1 \pm i.$$

Since α and β are the roots of $t^2 - 2t + 2 = 0$, we have

$$\alpha = 1 + i \text{ and } \beta = 1 - i.$$

$\therefore \quad (x+\alpha)^n - (x+\beta)^n = (x+1+i)^n - (x+1-i)^n.$

Put $\quad x+1 = r \cos \phi, \ 1 = r \sin \phi,$ so that

$\cot \phi = x + 1.$

Now $\quad (x+\alpha)^n - (x+\beta)^n = (\cos \phi + i \sin \phi)^n - (\cos \phi - i \sin \phi)^n$

$= r^n [(\cos \phi + i \sin \phi)^n - (\cos \phi - i \sin \phi)^n]$

$= r^n [\cos n\phi + i \sin n\phi - \cos n\phi + i \sin n\phi]$

$= 2i \, r^n \sin n\phi.$

Also $\alpha - \beta = (1+i) - (1-i) = 2i.$

$\therefore \quad \dfrac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = \dfrac{2i r^n \sin n\phi}{2i} = r^n \sin n\phi \qquad \dots(1)$

Hence $\quad \dfrac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = r^n \sin n\phi$

$= \dfrac{\sin n\phi}{\sin^n \phi}. \qquad (\because \ 1 = r \sin \phi \text{ gives } r^n = 1/\sin^n \phi).$

Putting $x = 1$ in the given equation, we have

$$\frac{\sin n\phi}{\sin^n \phi} = \frac{\sin n\theta}{\sin^n \theta}.$$

Putting this value in the given equation, we have

Therefore $\quad \phi = \theta$ so that $\cot \phi = \cot \theta,$

or $\qquad x + 1 = \cot \theta$

or $\qquad x = \cot \theta - 1.$ \qquad [$\because$ from (1), $\cot \phi = x+1$]

Ex. 35. *If* $x = \cos \theta + i \sin \theta$ *and* $\sqrt{(1-c^2)} = nc - 1,$
show that $1 + c \cos \theta = \dfrac{c}{2n} (1+nx)\left\{1 + \dfrac{n}{x}\right\}.$

(**Meerut 1989, 84S; Agra 79; Rohilkhand 78; Kanpur 81**)

Sol. If $x = \cos \theta + i \sin \theta$, we have

$1/x = 1/(\cos \theta + i \sin \theta) = (\cos \theta + i \sin \theta)^{-1} = \cos \theta - i \sin \theta.$

Also $\sqrt{(1-c^2)} = (nc - 1),$

$\Rightarrow 1 - c^2 = (nc-1)^2,$

$\Rightarrow 1 - c^2 = n^2c^2 - 2nc + 1,$ \quad squaring both sides

$\Rightarrow c^2 (n^2+1) = 2nc \Rightarrow c (1+n^2)/2n = 1.$

Now the R.H.S $= \dfrac{c}{2n} (1+nx)\left(1 + \dfrac{n}{x}\right)$

$= \dfrac{c}{2n} \left(1+nx + \dfrac{n}{x} + n^2\right) = \dfrac{c}{2n}(1+n^2) + \dfrac{c}{2n} \cdot n\left(x + \dfrac{1}{x}\right)$

$\qquad \qquad \qquad \qquad \qquad \qquad [\because \ c (1+n^2)/2n = 1]$

$= 1 + \tfrac{1}{2}c [(\cos \theta + i \sin \theta) + (\cos \theta - i \sin \theta)]$

$= 1 + \tfrac{1}{2}c.2 \cos \theta = 1 + c \cos \theta =$ the L.H.S.

Ex. 36. *Show that if* $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots,$
where n *is a positive integer, then*

(i) $\quad C_0 - C_2 + C_4 - \dots = 2^{n/2} \cos (n\pi/4).$

(**Meerut 1989, 79; Kanpur 81**)

(ii) $\quad C_1 - C_3 + C_5 - \dots = 2^{n/2} \sin (n\pi/4).$

(iii) $\quad C_0 + C_4 + C_8 + \dots = 2^{n-2} + (\sqrt 2)^{n-2} . \cos (n\pi/4).$

(**Lucknow 1980, 78**)

Sol. Let $\quad 1 + i = r (\cos \theta + i \sin \theta).$

Equating real and imaginary parts, we have

$$r \cos \theta = 1 \text{ and } r \sin \theta = 1$$

$\therefore \quad r = \sqrt{(1+1)} = \sqrt 2$ and $\tan \theta = 1$ i.e., $\theta = \tfrac{1}{4}\pi.$

so that

$(1+i)^n = r^n (\cos \theta + i \sin \theta)^n = r^n (\cos n\theta + i \sin n\theta)$

$= (\sqrt 2)^n (\cos \tfrac{1}{4}n\pi + i \sin \tfrac{1}{4}n\pi),$ \quad putting for r and $\theta.$ \qquad $\dots(1)$

Also we have

$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots$ \qquad $\dots(2)$

Putting $x = i$ in (2), we have

$(1+i)^n = C_0 + C_1 i - C_2 - C_3 i + C_4 + C_5 i - C_6 - C_7 i + \dots$

$= (C_0 - C_2 + C_4 - \dots) + i (C_1 - C_3 + C_5 - \dots).$ \qquad $\dots(3)$

$\qquad [\because \ i^2 = -1, \ i^3 = -i, \ i^4 = -1, \text{ and so on}]$

Equating the values of $(1+i)^n$ from (1) and (3), we have

$(C_0 - C_2 + C_4 - \dots) + i (C_1 - C_3 + C_5 - \dots)$

$= (\sqrt 2)^n [\cos \tfrac{1}{4}n\pi + i \sin \tfrac{1}{4}n\pi].$