The eq^n (5) is exactly similar to the eq^n of motion for a two body problem under inverse square law.

Hence the motion of mass m_1 reduces to a motion similar to the motion of two body problem as if it was unit mass under the attraction of M_1 and were placed at the centre of mass.

Similar results can be found for the other two masses ~~there~~ Hence the orbit of the masses is a conic section i.e an ellipse, parabola, hyperbola depending upon the initial velocity.

The restricted three body problem :

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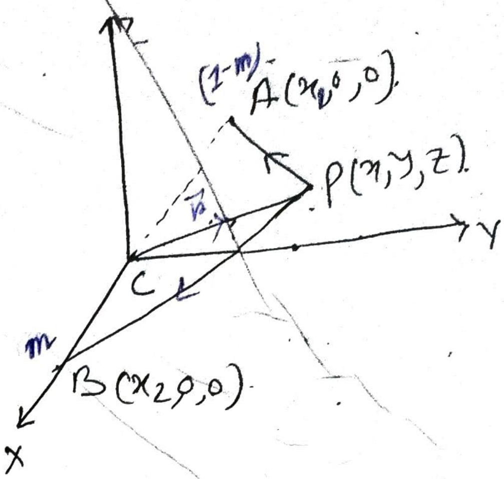
Here two bodies of finite mass revolve around one another in circular orbits, and a third body of infinitessinged mass moves in their field. This situation is approximately

realized in many instances in the solar system.

Let the origin C be at the centre of mass of the two finite masses and take axes rotating with the masses such that they lie along the x-axis. Let us take the unit of mass to be the ~~same~~ sum of their masses and let the ~~spe~~ separate masses be m and $(1-m)$, where $m \le \frac{1}{2}$.



The axes will be rotating with constant angular velocity ω, say and the bodies will be fixed at $(x_2,0,0)$ and $(x_1,0,0)$, where x_1 is negative.

Let $P(x,y,z)$ be the possition of the infinitesimal mass. Let $\vec{r} = \vec{CP}$

$\vec{v}$ = velocity of P relative to the moving system $Cxyz$

$= \dot{\vec{r}}\,]_M$

$\vec{a}$ = acceleration of P relative to the moving system $Cxyz$

$= \ddot{\vec{r}}\,]_M$

The eqn of motion of the infinitessimal mass relative to the fixed system at C is

$$\left.\frac{d^2\vec{r}}{dt^2}\right]_F = \frac{K^2 m}{PB^2}\cdot\frac{\vec{PB}}{PB} + \frac{K^2(1-m)}{PA^2}\cdot\frac{\vec{PA}}{PA}$$

$$\Rightarrow \left.\frac{d^2\vec{r}}{dt^2}\right]_M + \left.\frac{d\vec{\omega}}{dt}\times\vec{r}\right]_M + 2\vec{\omega}\times\left.\frac{d\vec{r}}{dt}\right]_M + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$= -\frac{K^2 m}{BP^2}\frac{\vec{BP}}{BP} - \frac{K^2(1-m)}{AP^2}\cdot\frac{\vec{AP}}{AP}$$

$$= -\frac{m\vec{\rho_2}}{\rho_2^3} - \frac{(1-m)\vec{\rho_1}}{\rho_1^3}, \quad [\text{taking } r=1,\ \vec{AP}=\rho_1,\ \vec{BP}=\vec{\rho_2}]$$

$$\Rightarrow \vec{a} + 2\vec{\omega}\times\vec{v} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) = -\frac{(1-m)}{\rho_1^3}\vec{\rho_1} - \frac{m}{\rho_2^3}\vec{\rho_2} \to (1)$$

where $\vec{\omega} = n\hat{k}$, the constant angular velocity of the moving system relative to the fixed axis. We have $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$, where $\hat{i}, \hat{j}, \hat{z}$ be the unit vectors along $\vec{CX}, \vec{CY}, \vec{CZ}$ resp.

$$\vec{v} = \dot{\vec{r}} = \hat{i}\dot{x} + \hat{j}\dot{y} + \hat{k}\dot{z}$$

$$\& \quad \vec{a} = \ddot{\vec{r}} = \hat{i}\ddot{x} + \hat{j}\ddot{y} + \hat{k}\ddot{z}$$

$$\vec{\omega} \times \vec{v} = n\hat{k} \times (\hat{i}\,\dot{x} + \hat{j}\,\dot{y} + \hat{k}\,\dot{z})$$
$$= n\dot{x}\,\hat{j} - n\dot{y}\,\hat{i}$$

and $\vec{\omega} \times \vec{r} = n\hat{k} \times (x\hat{i} + y\hat{j} + z\hat{k})$

$$= nx\hat{j} - ny\hat{i}$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = n\hat{k} \times (nx\hat{j} - ny\hat{i})$$
$$= -n^2 x\hat{i} - n^2 y\hat{j}$$
$$= -n^2 (x\hat{i} + y\hat{j}).$$

$$\vec{\rho_1} = (x - x_1)\hat{i} + y\hat{j} + z\hat{k}$$
$$\vec{\rho_2} = (x - x_2)\hat{i} + y\hat{j} + z\hat{k}$$

The cartesian eqn of motion are

$$\ddot{x} - 2n\dot{y} = n^2 x - \frac{(1-m)}{\rho_1^3}(x - x_1) - \frac{m}{\rho_2^3}(x - x_2) \rightarrow ②$$

$$\ddot{y} + 2n\dot{x} = n^2 y - \frac{(1-m)}{\rho_1^3}\,y - \frac{m}{\rho_2^3}\,y \rightarrow ③$$

$$\ddot{z} = -\frac{(1-m)}{\rho_1^3}\,z - \frac{m}{\rho_2^3}\,z \rightarrow ④.$$

Now, by keplers third law,

$$T = \frac{2\pi}{\sqrt{\mu}}\,a^{3/2}$$
$$= \frac{2\pi}{K\sqrt{m_1 + m_2}}\,a^{3/2}$$

$$n = \frac{2\pi}{T} = \frac{k\sqrt{m_1+m}}{a^{3/2}}$$

Let the unit of distance be $(-x_1+x_2)$ [i.e. $a=1$]

and let the unit of time be such as to make $k=1$.

$n = 1$, [$\because k=1$ & $m_1+m_2=1$]

$\therefore$ the eqn of motion becomes,

$$\ddot{x} - 2\dot{y} = x - \frac{(1-m)}{\rho_1^3}(x-x_1) - \frac{m}{\rho_2^3}(x-x_2) \rightarrow (6)$$

$$\ddot{y} + 2\dot{x} = y - \frac{(1-m)}{\rho_1^3}y - \frac{m}{\rho_2^3}y \rightarrow (7)$$

$$\ddot{z} = -\frac{(1-m)}{\rho_1^3}z - \frac{m}{\rho_2^3}z \rightarrow (8)$$?

The general problem of determining the motion of the infinitesimal mass is, therefore one requiring six integrals for its complete solutions.

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Now, let us consider a function U defined by

$$U = \frac{1}{2}(x^2+y^2) + \frac{1-m}{\rho_1} + \frac{m}{\rho_2} \rightarrow (9)$$