Linear Differential equation

Any eqn which is of the form

$$\frac{dy}{dx} + Py = q$$ where P and q are some functions of x or constant is called linear differential equation of the first order. [Note: Linear because the independent variable and its derivative has no exponent. So no y^2, y^3, $\sqrt{y}$, $\sin y$, $\log y$ etc. just plain y]

Solve $\frac{dy}{dx} + Py = q$

Multiplying both sides by $e^{\int P dx}$ we get

$$\frac{dy}{dx} e^{\int P dx} + Py\, e^{\int P dx} = q e^{\int P dx}$$

$$\Rightarrow \frac{d}{dx}\left(y e^{\int P dx}\right) = q e^{\int P dx}$$

$$\Rightarrow \int d\left(y e^{\int P dx}\right) = \int q e^{\int P dx} dx$$

$$\Rightarrow y e^{\int P dx} = \int q e^{\int P dx} dx + C,$$ which is the reqd solⁿ.

Note: (1) Here $e^{\int P dx}$ is called an integrating factor, (I.F). Then $I.F = e^{\int P dx}$ and the solⁿ is

$$\boxed{y\,(I.F) = \int q\,(I.F)\,dx + C}$$

Note 2. The equation $\frac{dx}{dy} + Px = q$ is also called linear differential equation of 1st order where p and q are some functions of y or constant. Here $I.F = e^{\int p\,dy}$ and the sol^n is $x(I.F) = \int q(I.F)\,dy + c$

Example (1) $\frac{dy}{dx} + y = x$, which is linear in y

$I.F = e^{\int p\,dx} = e^{\int 1 \cdot dx} = e^{x}$ and

the sol^n is: $y e^{x} = \int x e^{x}\,dx + c$

$\Rightarrow y e^{x} = x e^{x} - \int 1 \cdot e^{x}\,dx + c$ [Integrating by parts]

$\Rightarrow y e^{x} = x e^{x} - e^{x} + c$

$\Rightarrow y = x - 1 + \frac{c}{e^{x}}$ Ans

or $y = x - 1 + c e^{-x}$

Ex (2) $\frac{dy}{dx} + xy = x$ (can be done by variable separation method)

Here $\frac{dy}{dx} + xy = x$, which is linear in y

$I.F = e^{\int p\,dx} = e^{\int x\,dx} = e^{\frac{x^2}{2}}$

And the sol^n is

$y e^{\int p\,dx} = \int q e^{\int p\,dx}\,dx + c$

$\Rightarrow y e^{x/2} = \int x e^{x^2/2}\,dx + c$

$$\Rightarrow ye^{\frac{x^2}{2}} = \int e^{\frac{x^2}{2}} d\left(\frac{x^2}{2}\right) + c \qquad \Big|\ x\,dx = d\left(\frac{x^2}{2}\right)$$

$$\Rightarrow ye^{\frac{x^2}{2}} = e^{x^2/2} + c$$

$$\Rightarrow y = 1 + ce^{-x^2/2}$$

Ex(3) $\frac{dy}{dx} + \left(\frac{1-2x}{x^2}\right)y = 1$, Which is linear in Y

$$\text{I.F.} = e^{\int P\,dx} = e^{\int \frac{1-2x}{x^2} dx} = e^{\int\left(\frac{1}{x^2} - \frac{2}{x}\right)dx}$$

$$= e^{-\frac{1}{x} - 2\log x}$$

$$= e^{-\frac{1}{x}} \cdot e^{-2\log x}$$

$$= e^{-\frac{1}{x}} \cdot e^{\log x^{-2}}$$

$$= e^{-\frac{1}{x}} \cdot \frac{1}{x^2} = \frac{e^{-\frac{1}{x}}}{x^2}$$

Since $e^{\log x} = x$

$e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$

And the sol^n is

$$y\frac{e^{-\frac{1}{x}}}{x^2} = \int 1 \cdot \frac{e^{-\frac{1}{x}}}{x^2} dx + c$$

$$\Rightarrow ye^{-\frac{1}{x}}\frac{1}{x^2} = \int e^{-\frac{1}{x}} d\left(-\frac{1}{x}\right) dx + c \qquad \Big|\ d\left(-\frac{1}{x}\right) = -\left(-\frac{1}{x^2}\right) = \frac{1}{x^2}$$

$$\Rightarrow ye^{-\frac{1}{x}} \cdot \frac{1}{x^2} = e^{-\frac{1}{x}} + c$$

$$\Rightarrow \frac{y}{x^2} = 1 + ce^{1/x}$$

$$\Rightarrow y = x^2 + \frac{c}{x^2}e^{\frac{1}{x}}$$

Ans

First order linear differential equation with constant coefficient and constant term

In a linear differential eqn of first order $\frac{dy}{dx} + Py = q$, where p and q are constants,

the equation becomes

$$\frac{dy}{dx} + ay = b \longrightarrow (1)$$

Case I When constant term $b = 0$, The equation becomes

$$\frac{dy}{dx} + ay = 0$$

$$\Rightarrow \frac{dy}{dx} = -ay$$

$$\Rightarrow \int \frac{dy}{y} = \int -a\,dx$$

$$\Rightarrow \log y = -ax + c$$

$$\Rightarrow y = e^{-ax+c} = e^{-ax} \cdot e^{c} = Ae^{-ax}, \quad A = e^{c}$$

Thus, $y(x) = Ae^{-ax}$ (i) is the general soln of the eqn.

Putting $x = 0$ in (i) we get

$$y(0) = A e^{0} = A \cdot 1 = A.$$

From (i), $y(x) = y(0)\, e^{-ax} \longrightarrow$ (ii) which is the definite solution.

Case II when $b \neq 0$, Then the solution consist of two parts – complementary sol^n and particular sol^n. The sum of theses two parts gives the general solution.

The complementary sol^n, y_c is the sol^n of the part:
$$\frac{dy}{dx} + ay = 0.$$

Therefore, $y_c = Ae^{-ax}$

To find out the particular sol^n, y_p we put.
$$y = c \quad , c \text{ is constant}.$$

[Note: Since particular sol^n or particular integral is any particular sol^n, we can consider it to be a constant]

when $y = c$, $\frac{dy}{dx} = 0$

Then equa (1) becomes, $ay = b$

$$\Rightarrow y = \frac{b}{a}, \; a \neq 0$$

$$\text{ie } y_p = \frac{b}{a}$$

So the complete or general sol^n is given by
$$y = y_c + y_p$$
$$\text{ie } y(x) = Ae^{-ax} + \frac{b}{a}, \; (a \neq 0) \qquad (2)$$

Now, taking the initial condition that is putting $x=0$ in (2)

$$y(0) = A.e^{0} + \frac{b}{a}$$
$$= A + \frac{b}{a}$$
$$\Rightarrow A = y(0) - \frac{b}{a}$$

Therefore, the definite solution is given by

$$y(x) = \left[y(0) - \frac{b}{a}\right] e^{-ax} + \frac{b}{a}, \quad (a \neq 0)$$

Example (1) solve $\frac{dy}{dx} + 5y = 10$, given that when $x=0$, $y=6$.
(1)

Solⁿ: This is a first order differential equation with constant co-efficient and constant term. Hence the solution consists of two parts — complementary function (y_c) and the particular integral (y_p).

The complementary f^n is given by

$$\frac{dy}{dx} + 5y = 0$$
$$\Rightarrow \frac{dy}{dx} = -5y$$
$$\Rightarrow \int \frac{dy}{y} = \int -5\, dx$$

$\Rightarrow \log y = -5x + c$

$\Rightarrow y = e^{-5x+c} = e^{-5x} \cdot e^{c} = Ae^{-5x}$, $e^{c} = A$.

$\therefore y_c = Ae^{-5x}$

To find particular integral y_p, we put

$y = c$, c is constant

$\therefore \frac{dy}{dx} = 0$

Then eqn ① becomes

$5y = 10$

$\Rightarrow y = \frac{10}{5} = 2$

ie $y_p = 2$

Therefore, the general solution is

$y = y_c + y_p$

or $y(x) = Ae^{-5x} + 2 \longrightarrow (2)$

Now, given $y(0) = 6$ when $x = 0$.

$\therefore$ From (2), $y(0) = Ae^{-5 \cdot 0} + 2$

$\Rightarrow 6 = A \cdot e^{0} + 2 = A \cdot 1 + 2 = A + 2$

$\Rightarrow A = 6 - 2 = 4$

$\therefore$ The reqd final soln is: $\underline{y(x) = 4e^{-5x} + 2}$