Theorem 2.8.1. **(De Moivre's theorem)**

When n is an integer, positive or negative, and θ is a real number

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta;$$

when n is a fraction, positive or negative, and θ is a real number

$\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

Proof. **Case I.** Let n be a positive integer.

The theorem holds for $n = 1$, since $(\cos +i\sin\theta)^1 = \cos\theta + i\sin\theta = \cos 1\theta + i\sin 1\theta$.

Let us assume that the theorem holds for $n = m$, where m is a positive integer.

Then $(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$.

Therefore $(\cos\theta + i\sin\theta)^{m+1} = (\cos m\theta + i\sin m\theta)(\cos\theta + i\sin\theta)$

$= (\cos m\theta\cos\theta - \sin m\theta\sin\theta) + i(\cos m\theta\sin\theta + \sin m\theta\cos\theta)$

$= \cos(m+1)\theta + i\sin(m+1)\theta$.

This shows that the theorem holds for $n = m + 1$ if we assume it to hold for $n = m$. And the theorem holds for $n = 1$.

Therefore, by the principle of induction, the theorem holds for all positve integers n.

Case 2. Let n be a negative integer.

Let $n = -m$, where m is positive integer.

Now $(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1}{\cos m\theta + i\sin m\theta}, \text{ by case 1}$$

$$= \frac{(\cos m\theta - i\sin m\theta)}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

$$= \cos m\theta - i\sin m\theta$$

$$= \cos(-n)\theta - i\sin(-n)\theta$$

$$= \cos n\theta + i\sin n\theta.$$

Case 3. Let n be a fraction, positive or negative.

Let $n = p/q$ where p, q are integers and $q > 1, p$ may be positive or negative.

Since $(\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q})^q = \cos p\theta + i\sin p\theta$, by case 1

$= (\cos\theta + i\sin\theta)^p$, by case 1 or case 2,

it follows that $\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}$ is one of the values of $(\cos\theta + i\sin\theta)^{p/q}$, i.e., $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$. ▱

Corollary. When n is a positive or a negative integer and θ is a real number

$$(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta;$$

when n is a fraction and θ is a real number

$(\cos n\theta - i\sin n\theta)$ is one of the values of $(\cos\theta - i\sin\theta)^n$.

This follows from the relation

$$(\cos\theta + i\sin\theta)^{-1} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta.$$

Note. The generalised form of De Moivre's theorem states that if n be a real number and θ is real $(\cos n\theta + i\sin n\theta)$ is a value of $(\cos\theta + i\sin\theta)^n$.

Worked Examples.

1. Use De Moivre's theorem to prove $\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$.

We have $\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$

$$\begin{aligned} &= \cos 5\theta + 5c_1 \cos^4\theta i \sin\theta + 5c_2 \cos^3\theta i^2 \sin^2\theta \\ &+ 5c_3 \cos^2\theta i^3 \sin^3\theta + \cdots + i^5 \sin^5\theta \\ &= (\cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta) \\ &+ i(5\cos^4\theta \sin\theta - 10\cos^2\theta \sin^3\theta + \sin^5\theta). \end{aligned}$$

Therefore $\cos 5\theta = \cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta)$,
$\sin 5\theta = 5\cos^4\theta \sin\theta - 10\cos^2\theta \sin^3\theta + \sin^5\theta$.

$$\begin{aligned} \text{Hence } \tan 5\theta &= \frac{5\cos^4\theta \sin\theta - 10\cos^2\theta \sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta} \\ &= \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}. \end{aligned}$$

2. Prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

From Example 1, $\cos 5\theta = \cos^5\theta - 10\cos^3 \sin^2\theta + 5\cos\theta \sin^4\theta$

$$\begin{aligned} &= t^5 - 10t^3(1-t^2) + 5t(1-t^2)^2, \text{ where } t = \cos\theta \\ &= t^5 - 10t^3(1-t^2) + 5t(1-2t^2+t^4) \\ &= 16t^5 - 20t^3 + 5t \\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta. \end{aligned}$$

3. Expand $\cos^7\theta$ in a series of cosines of multiples of θ.

Let $x = \cos\theta + i\sin\theta$.

Then $x^n + \frac{1}{x^n} = 2\cos n\theta$, $\quad x + \frac{1}{x} = 2\cos\theta$.

Now $(2\cos\theta)^7 = (x + \frac{1}{x})^7$

$= x^7 + 7x^5 + 21x^3 + 35x + 35.\frac{1}{x} + 21.\frac{1}{x^3} + 7.\frac{1}{x^5} + \frac{1}{x^7}$

$= (x^7 + \frac{1}{x^7}) + 7(x^5 + \frac{1}{x^5}) + 21(x^3 + \frac{1}{x^3}) + 35(x + \frac{1}{x})$

$= 2\cos 7\theta + 7.2\cos 5\theta + 21.2\cos 3\theta + 35.2\cos\theta$.

Therefore $\cos^7\theta = \frac{1}{64}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$.

4. Expand $\sin^7\theta$ in a series of sines of multiples of θ.

Let $x = \cos\theta + i\sin\theta$.

Then $x^n - \frac{1}{x^n} = 2i \sin n\theta$, $\quad x - \frac{1}{x} = 2i \sin \theta$.

Now $(2i \sin \theta)^7 = (x - \frac{1}{x})^7$

$= x^7 - 7x^5 + 21x^3 - 35x + 35.\frac{1}{x} - 21.\frac{1}{x^3} + 7\frac{1}{x^5} - \frac{1}{x^7}$

$= (x^7 - \frac{1}{x^7}) - 7(x^5 - \frac{1}{x^5}) + 21(x^3 - \frac{1}{x^3}) - 35(x - \frac{1}{x})$

$= 2i \sin 7\theta - 7.2i \sin 5\theta + 21.2i \sin 3\theta - 35.2i \sin \theta$.

Therefore $\sin^7 \theta = -\frac{1}{64}(\sin 7\theta - 7 \sin 5\theta + 21 \sin 3\theta - 35 \cos \theta)$.

5. Expand $\sin^4 \theta \cos^2 \theta$ in a series of cosines of multiples of θ.

Let $x = \cos \theta + i \sin \theta$.

Then $x + \frac{1}{x} = 2 \cos \theta$, $\quad x - \frac{1}{x} = 2i \sin \theta$.

$x^n + \frac{1}{x^n} = 2 \cos n\theta$, $\quad x^n - \frac{1}{x^n} = 2i \sin n\theta$.

Now $(2i \sin \theta)^4 (2 \cos \theta)^2 = (x - \frac{1}{x})^4 (x + \frac{1}{x})^2$

$= (x^2 - \frac{1}{x^2})^2 (x - \frac{1}{x})^2$

$= (x^4 - 2 + \frac{1}{x^4})(x^2 - 2 + \frac{1}{x^2})$

$= (x^6 + \frac{1}{x^6}) - 2(x^4 + \frac{1}{x^4}) - (x^2 + \frac{1}{x^2}) + 4$

$= 2 \cos 6\theta - 4 \cos 4\theta - 2 \cos 2\theta + 4$.

Therefore $\sin^4 \theta \cos^2 \theta = \frac{1}{32}(\cos 6\theta - 2 \cos 4\theta - \cos 2\theta + 2)$.