Homogeneous differential Equation

def^n Any equation which is of the form

$\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$ where $f_1(x,y)$ and

$f_2(x,y)$ are homo. functions of same dimension (or degree) is called a homo. diff. equation.

[Procedure: Putting $y = vx$, $\frac{dy}{dx} = v.1 + x.\frac{dv}{dx}$ etc]

Example 1: Solve the differential equation:

$$(x^2+y^2)dx - 2xy\,dy = 0$$

$$\Rightarrow (x^2+y^2)dx = 2xy\,dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2+y^2}{2xy}$$

Let $y = vx$

$\frac{dy}{dx} = v.1 + x\frac{dv}{dx}$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2x.vx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x^2(1+v^2)}{2vx^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{2v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2-2v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\Rightarrow \int\frac{2v\,dv}{1-v^2} = \int\frac{dx}{x}$$

$$\Rightarrow -\log(1-v^2) = \log x + \log c$$

Let $1-v^2 = u$

$\therefore -2v\,dv = du$

$\therefore \int\frac{2v}{1-v^2} = -\int\frac{du}{u}$

$= -\log u$

$= -\log(1-$

$\Rightarrow \log(1-v^2) = -\log x - \log c$

$\Rightarrow \log(1-v^2) + \log x + \log c = 0$

$\Rightarrow \log(1-v^2)xc = 0$

$\Rightarrow cx(1-v^2) = e^0 = 1$

$\Rightarrow cx\left\{1-\left(\frac{y}{x}\right)^2\right\} = 1$ | $y = vx$ $\Rightarrow \frac{y}{x} = v$

$\Rightarrow cx\,\frac{x^2-y^2}{x^2} = 1$

$\Rightarrow \frac{c(x^2-y^2)}{x} = 1$

$\Rightarrow c(x^2-y^2) = x$ Ans

Ex(2) $(x^2+y^2)\,dy = xy\,dx$

$\Rightarrow \frac{dy}{dx} = \frac{xy}{x^2+y^2}$ | Let $y = vx$ $\therefore \frac{dy}{dx} = v.1 + x\frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{x.vx}{x^2+v^2x^2}$

$\Rightarrow x\frac{dv}{dx} = \frac{vx^2}{x^2(1+v^2)} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{v}{1+v^2} - v$ | $v - v - v^3$

$\Rightarrow x\frac{dv}{dx} = \frac{v-v-v^3}{1+v^2} = \frac{-v^3}{1+v^2}$

$\Rightarrow \frac{1+v^2}{v^3}\,dv = -\frac{dx}{x}$

$$\Rightarrow \int \frac{1+v^2}{v^3}\,dv = -\int \frac{dx}{x}$$

$$\Rightarrow \int \frac{dv}{v^3} + \int \frac{1}{v}\,dv = -\int \frac{dx}{x}$$

$$\Rightarrow -\frac{1}{2v^2} + \log v = -\log x + \log c$$

$$\Rightarrow \log v + \log x - \log c = \frac{1}{2v^2}$$

$$\Rightarrow \log\left(\frac{vx}{c}\right) = \frac{1}{2v^2}$$

$$\Rightarrow \frac{vx}{c} = e^{\frac{1}{2v^2}}$$

$$\Rightarrow \frac{y}{x}\cdot\frac{x}{c} = e^{\frac{1}{2(y/x)^2}}$$

$$\Rightarrow \frac{y}{c} = e^{\frac{x^2}{2y^2}}$$

$$\Rightarrow y = c\,e^{\frac{x^2}{2y^2}} \quad \text{Ans}$$

$$\int v^{-3}\,dv = \frac{v^{-3+1}}{-3+1} = \frac{v^{-2}}{-2} = \frac{-1}{2v^2}$$

$$y = vx$$
$$\Rightarrow \frac{y}{x} = v$$

Ex (3) $\quad \dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{y^2}{x^2}$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2}{x^2} - \frac{y}{x} = \frac{y^2 - xy}{x^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v^2x^2 - v.x^2}{x^2}$$

Let $y = vx$

$$\frac{dy}{dx} = v.1 + x\frac{dv}{dx}$$

$$\Rightarrow v + \frac{x\,dv}{dx} = v^2 - v$$

$$\Rightarrow x\frac{dv}{dx} = v^2 - v - v = v^2 - 2v$$

$$\Rightarrow \frac{dv}{v^2 - 2v} = \frac{dx}{x}$$

$$= \int \frac{dv}{v^2-2v} = \int \frac{dx}{x}$$

$$\frac{1}{v^2-2v} = \frac{1}{v(v-2)} = \frac{1}{2}\left(\frac{1}{v-2} - \frac{1}{v}\right)$$

$$\Rightarrow \int \frac{1}{2}\left(\frac{1}{v-2} - \frac{1}{v}\right) dv = \int \frac{dx}{x}$$

$$v - \cancel{v} + 2 = +2$$

$$\Rightarrow \frac{1}{2}\int \frac{dv}{v-2} - \frac{1}{2}\int \frac{dv}{v} = \int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\log(v-2) - \frac{1}{2}\log v = \log x + \log c$$

$$\Rightarrow \log(v-2) - \log v = 2\log x + 2\log c$$

$$\Rightarrow \log\left(\frac{v-2}{v}\right) = \log x^2 + \log c^2$$

$$\Rightarrow \log\left(\frac{v-2}{v}\right) = \log c^2 x^2$$

$$\Rightarrow \frac{v-2}{v} = c^2 x^2$$

$$y = vx \Rightarrow \frac{y}{x} = v$$

$$\Rightarrow \frac{\frac{y}{x} - 2}{\frac{y}{x}} = c^2 x^2$$

$$\Rightarrow \frac{x}{y}\left(\frac{y-2x}{x}\right) = c^2 x^2$$

$$\Rightarrow \frac{y-2x}{y} = c^2 x^2$$

$$\Rightarrow y - 2x = Kyx^2, \quad c^2 = K$$

Ans

Ex(4) $(x^2-y^2)dx + xy\,dy = 0$

$\Rightarrow (x^2-y^2)dx = -xy\,dy$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{x^2-y^2}{xy}$

$\Rightarrow v + x\dfrac{dv}{dx} = -\dfrac{x^2-v^2x^2}{x\cdot v\cdot x}$

Let $y = vx$

$\therefore \dfrac{dy}{dx} = v\cdot 1 + x\dfrac{dv}{dx}$

$\Rightarrow v + x\dfrac{dv}{dx} = -\dfrac{x^2(1-v^2)}{vx^2}$

$\Rightarrow v + x\dfrac{dv}{dx} = \dfrac{v^2-1}{v}$

$\Rightarrow x\dfrac{dv}{dx} = \dfrac{v^2-1}{v} - v = \dfrac{v^2-1-v^2}{v} = \dfrac{-1}{v}$

$\Rightarrow v\,dv = -\dfrac{dx}{x}$

$\Rightarrow \int v\,dv = -\int \dfrac{dx}{x}$

$\Rightarrow \dfrac{v^2}{2} = -\log x + \log c$

$\Rightarrow \dfrac{v^2}{2} = \log\dfrac{c}{x}$

$\Rightarrow \dfrac{c}{x} = e^{v^2/2} = e^{\frac{(y/x)^2}{2}} = e^{\frac{y^2}{2x^2}}$

$\Rightarrow c = x e^{y^2/2x^2}$ Ans

Ex (5)

$$\left(\frac{x}{y} + \frac{y}{x}\right)\frac{dy}{dx} = 1$$

$$\Rightarrow \left(\frac{x^2+y^2}{xy}\right)\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{xy}{x^2+y^2}$$ [Consult (Ex 2)]

x

Ex (6) $x^2 dy + y(x+y)dx = 0$

$\Rightarrow x^2 dy = -y(x+y)dx$

$\Rightarrow \frac{dy}{dx} = \frac{-y(x+y)}{x^2} = \frac{-yx - y^2}{x^2}$

Let $y = vx$

$\frac{dy}{dx} = v \cdot 1 + x\frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{-vx^2 - v^2x^2}{x^2}$

$\Rightarrow x\frac{dv}{dx} = \frac{-\cancel{x^2}(v+v^2)}{\cancel{x^2}} - v$

$\Rightarrow x\frac{dv}{dx} = -v - v^2 - v = -v^2 - 2v$

$\Rightarrow \int \frac{dv}{-v^2-2v} = \int \frac{dx}{x}$

$\Rightarrow -\int \frac{dv}{v(v+2)} = \int \frac{dx}{x}$

$\frac{1}{v(v+2)} = \frac{1}{2}\left(\frac{1}{v} - \frac{1}{v+2}\right)$

$\Rightarrow -\int \frac{1}{2}\left(\frac{1}{v} - \frac{1}{v+2}\right)dv = \int \frac{dx}{x}$

$v + 2 - v = 2$

$\Rightarrow -\frac{1}{2}\int \frac{dv}{v} + \frac{1}{2}\int \frac{dv}{v+2} = \int \frac{dx}{x}$

$\Rightarrow -\frac{1}{2}\log v + \frac{1}{2}\log(v+2) = \log x + \log c$

$\Rightarrow -\log v + \log(v+2) = 2\log x + 2\log c$

$\Rightarrow \log\left(\frac{v+2}{v}\right) = \log x^2 c^2$

$\Rightarrow \frac{v+2}{v} = x^2 c^2$

$y = vx$

$\frac{y}{x} = v$

$\Rightarrow v + 2 = x^2 c^2 v$

$\Rightarrow \frac{y}{x} + 2 = x^2 c^2 \cdot \frac{y}{x}$

$\Rightarrow \frac{y + 2x}{x} = c^2 x y$

$\Rightarrow y + 2x = c^2 x^2 y$ Ans

(7) $x + y\frac{dy}{dx} = 2y$

$\Rightarrow y\frac{dy}{dx} = 2y - x$

$\Rightarrow \frac{dy}{dx} = \frac{2y-x}{y}$

Let, $y = vx$

$\frac{dy}{dx} = v.1 + x\frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{2vx - x}{vx}$

X

$\Rightarrow x\frac{dv}{dx} = \frac{\cancel{x}(2v-1)}{v\cancel{x}} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{2v-1-v^2}{v} = \frac{2v-v^2-1}{v}$

$\Rightarrow \int\frac{v\,dv}{2v-v^2-1} = \int\frac{dx}{x}$

$\Rightarrow -\int\frac{v}{(v-1)^2}dv = \int\frac{dx}{x}$

$\Rightarrow -\int\frac{v-1+1}{(v-1)^2}dv = \int\frac{dx}{x}$

$\Rightarrow -\int\frac{v-1}{(v-1)^2}dv - \int\frac{dv}{(v-1)^2} = \int\frac{dx}{x}$

$\Rightarrow -\int\frac{1\,dv}{(v-1)} - \int\frac{dv}{(v-1)^2} = \int\frac{dx}{x}$

$\Rightarrow -\log(v-1) - \int\frac{d(v-1)}{(v-1)^2} = \int\frac{dx}{x}$

$\Rightarrow -\log(v-1) + \frac{1}{v-1} = \log x + \log c$

$2v - v^2 - 1$
$= -(v^2 - 2v + 1)$
$= -(v-1)^2$

$\frac{v}{(v-1)^2} = \frac{v-1+1}{(v-1)^2}$
$= \frac{v-1}{(v-1)^2} + \frac{1}{(v-1)^2}$

Let
$v - 1 = t$
$dv = dt$

$\int\frac{dt}{t^2} = \int t^{-2}dt$
$= \frac{t^{-2+1}}{-2+1}$
$= \frac{-1}{t} = \frac{-1}{v-1}$

$\Rightarrow \frac{1}{v-1} = \log x + \log c + \log(v-1)$

$\Rightarrow \frac{1}{v-1} = \log cx(v-1)$

$\therefore cx(v-1) = e^{\frac{1}{v-1}}$

$\Rightarrow cx\left(\frac{y}{x}-1\right) = e^{\frac{1}{\frac{y}{x}-1}}$

$\Rightarrow c(y-x) = e^{\frac{x}{y-x}}$ Ans

Ex 8 $y^2 dx + (x^2+xy)dy = 0$

$\Rightarrow y^2 dx = -(x^2+xy)dy$

$\Rightarrow \frac{dy}{dx} = -\frac{y^2}{x^2+xy}$

Let $y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = -\frac{v^2x^2}{x^2+x\cdot vx}$

$\Rightarrow x\frac{dv}{dx} = \frac{-\cancel{x^2}v^2}{\cancel{x^2}(1+v)} - v = \frac{-v^2-v-v^2}{1+v}$

$\Rightarrow x\frac{dv}{dx} = \frac{-v-2v^2}{1+v} = \frac{-v(1+2v)}{1+v}$

$\Rightarrow -\int \frac{(1+v)\,dv}{v(1+2v)} = \int \frac{dx}{x}$

$\Rightarrow -\int \left[\frac{1}{v} - \frac{1}{1+2v}\right] dv = \int \frac{dx}{x}$

$\Rightarrow -\log v + \frac{1}{2}\log(1+2v) = \log x + \log c$

$\Rightarrow \frac{1}{2}\log(1+2v) = \log v + \log x + \log c$

$\Rightarrow \log(1+2v)^{\frac{1}{2}} = \log vxc$

$$\therefore (2v+1)^{\frac{1}{2}} = vxc$$

$$\Rightarrow 2v+1 = (vx)^2 c^2$$

$$\Rightarrow 2\frac{y}{x}+1 = y^2 c^2$$

$$\Rightarrow \frac{2y+x}{x} = c^2 y^2$$

$$\Rightarrow 2y+x = c^2 x y^2 \quad \text{Ans}$$