* LU Decomposition method or triangularization method:

In this method, the co-efficient matrix A is decompared to the product of a lower triangular matrix L and an upper triangular matrix U. i.e $A = LU$.

where,

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ \vdots & & & & \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{13} \\ 0 & u_{22} & \cdots & u_{2n} \\ 0 & 0 & \cdots & u_{3n} \\ \vdots & & & \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

where $l_{ii} = 1$, and $u_{ii} = 1$

To solve the system either we take,

$l_{ii} = 1, \ i = 1, 2, \ldots, n$

or $u_{ii} = 1, \ i = 1, 2, \ldots, n$

If $l_{ii} = 1$, then the method is called the Doolittle's method and if $u_{ii} = 1$, then the method is called the ~~Crouts method~~ Crouts method

In this method $Ax = b$ becomes, $LUx = b$.

Thus the system can be written as,

$Ux = z;$
$Lz = b.$

$$\left\{\begin{array}{l} Lz = b \\ Ux = z \end{array}\right.$$

The unknown z is determined by forward substitution method and x is determined by backward substitution method.

The LU method fails if ~~and~~ any of the diagonal elements l_{ii} and u_{ii} is zero.

Note! Not all matrices can be decomposed by the LU method but a positive definite matrix can always be decomposed.

Q. Solve by LU method

(i)
$$4x_1 + x_2 + x_3 = 4$$
$$x_1 + 4x_2 - 2x_3 = 4$$
$$3x_1 + 2x_2 - 4x_3 = 6$$

(ii)
$$x_1 + x_2 - x_3 = 2$$
$$2x_1 + 2x_2 + 5x_3 = -3$$
$$3x_1 + 2x_2 - 3x_3 = 6$$

Soln :

$$A = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} x = b$$

Now, at $l_{ii} = 1$, $i = 1, 2, 3$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 2 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+u_{12} & l_{31}u_{13}+u_{23} \end{bmatrix}$$

$\therefore u_{11} = 4$, $u_{12} = 1$, $u_{13} = 1$

$l_{21}u_{11} = 1 \Rightarrow l_{21} 4 = 1 \Rightarrow l_{21} = \frac{1}{4}$

$l_{21}u_{12} + u_{22} = 4 \Rightarrow \frac{1}{4} \times 1 + u_{22} = 4 \Rightarrow u_{22} = \frac{15}{4}$

$l_{21} + u_{23} = -2 \Rightarrow u_{23} = -9/4$

$l_{11} = l_{22} = l_{33} = 1$

$\boxed{u_{11} = 4}$, $\boxed{u_{12} = 1}$, $\boxed{u_{13} = 1}$ #

$l_{21} u_{11} = 1 \Rightarrow \boxed{l_{21} = 1/4}$

$l_{21} u_{12} + u_{22} = 4 \Rightarrow \boxed{u_{22} = 15/4}$

$l_{21} u_{13} + u_{23} = -2$

$\Rightarrow \boxed{u_{23} = -9/4}$.

$l_{31} u_{11} = 3 \Rightarrow \boxed{l_{31} = 3/4}$

$l_{31} u_{12} + l_{32} u_{22} = 2 \Rightarrow \boxed{l_{32} = 1/3}$

and $l_{31} u_{13} + l_{32} u_{23} + u_{33} = -4$

$\Rightarrow \boxed{u_{33} = -4}$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ 3/4 & \frac{1}{3} & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & 1 & 1 \\ 0 & \frac{15}{4} & -9/4 \\ 0 & 0 & -4 \end{bmatrix}$$

Now the given system $Ax = b$ can be written as $LUx = b$.

i.e the given system ~~becomes~~ becomes $Ux = Z$
$LZ = b$

Now, $LZ = b$ gives,
$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{3}{4} & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 6 \end{bmatrix}$$

So, $z_1 = 4$

$$\frac{1}{4}z_1 + z_2 = 4 \Rightarrow z_2 = 3$$

$$\frac{3}{4}z_1 + \frac{1}{3}z_2 + z_3 = 6 \Rightarrow z_3 = 2$$

Next, $Ux = z$ becomes,

$$\begin{bmatrix} 4 & 1 & 1 \\ 0 & 15/4 & -9/4 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

So,

$$4x_1 + x_2 + x_3 = 4 \Rightarrow x_1 = 1$$

$$\frac{15}{4}x_2 - \frac{9}{4}x_3 = 3 \Rightarrow x_2 = \frac{1}{2}$$

$$-4x_3 = 2 \Rightarrow x_3 = -\frac{1}{2}$$

$\therefore$ the required solution is, $x_1 = 1, x_2 = \frac{1}{2}, x_3 = -\frac{1}{2}$ #

H.W

Q: Solve

$$\begin{bmatrix} 2 & 1 & -4 & 1 \\ -4 & 3 & 5 & -2 \\ 1 & -1 & 1 & -1 \\ 1 & 3 & -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 2 \\ -1 \end{bmatrix}$$

by ~~Crout's method~~ Crout's method.

Q. Show that LU decomposition method fails to solve the system,

$$\begin{bmatrix} 2 & 1 & -2 \\ 4 & 2 & 3 \\ -6 & 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$$